Skew Brownian motion with dry friction: The Pugachev-Sveshnikov equation approach

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Introduction

- A. Einstein (1905), M. Smoluchowski (1906)

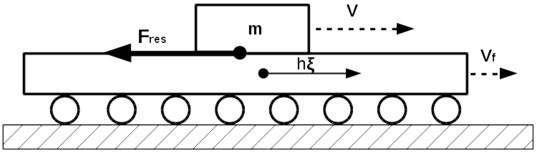
$$dX(t) = h\,dW(t) \tag{1}$$

- L. Ornstein, G. Uhlenbeck (1930)

$$dV(t) = -\alpha V(t)\,dt + h\,dW(t) \tag{2}$$

- T. Caughey, J. Dienes (1968) (velocity of Brownian motion with dry friction, Brownian motion with two-valued drift)

$$dV(t) = -\alpha\,\mathrm{sign}(V(t))\;dt + h\,dW(t) \tag{3}$$



Skew Brownian motion with dry friction

For any $\eta \in (-1, 1)$ there exist a unique strong solution to

$$\begin{cases} dX(t) = -2\mu \operatorname{sign}(X(t))\, dt + \eta\, dL_X^0(t) + \sqrt{2}\, dW(t),\ t > 0 \\ X(0) = 0 \end{cases} \tag{4}$$

$X(t)$ is the velocity of skew Brownian motion with dry friction

$W(t)$ is a standard Wiener process starting at zero
$L_X^0(t)$ is the symmetric local time of $X(t)$ at the level zero:

$$L_X^0(t) = \lim_{\varepsilon \to +0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(-\varepsilon, +\varepsilon)}(X(s))\, d[X]_s \tag{5}$$

$[X]_s = 2s$ is the quadratic variation of the continuous semimartingale $X(s)$

Positive half-line occupation time of $X(t)$:

$$\mathcal{I}(t) = \int_0^t \mathbb{1}_{(0, +\infty)}(X(s))\, ds \tag{6}$$

Equivalent SDE formulation

- Diffusion process with a singular drift

$$\begin{cases} dX(t) = -2\mu\,\mathrm{sign}(X(t))\,dt + \eta\,dL_X^0(t) + \sqrt{2}\,dW(t) \\ d\mathcal{I}(t) = \mathbb{1}_{(0,+\infty)}(X(s))\,ds \\ X(0) = 0,\ \mathcal{I}(0) = 0 \end{cases} \tag{7}$$

- Weak solution (distribution)
- Might as well use Fokker–Planck–Kolmogorov equation for the density, but how to deal with the local time or "sign" ?
- Pugachev equation for the characteristic function

General Pugachev equation

$$\begin{cases} d\boldsymbol{X}(t) = \boldsymbol{a}(t, \boldsymbol{X}(t))\, dt + \mathbf{H}(t, \boldsymbol{X}(t))\ d\boldsymbol{W}(t),\ t > 0 \\ X(0) = \boldsymbol{X}_0 \in \mathbb{R}^n \end{cases} \quad (8)$$

$\boldsymbol{W}(t)$ is a vector of independent Wiener processes

$E(\boldsymbol{z}, t) = E(\boldsymbol{z}, t) = \mathbb{E}\left[e^{i\boldsymbol{z}^T\boldsymbol{X}(t)}\right]$ is the characteristic function of $\boldsymbol{X}(t)$

Pugachev equation (Pugachev (1944)):

$$\begin{cases} \dfrac{\partial E(\boldsymbol{z}, t)}{\partial t} = \mathbb{E}\left[\Phi(\boldsymbol{z}|\boldsymbol{X}(t), t)e^{i\boldsymbol{z}^T\boldsymbol{X}(t)}\right] \\ E(\boldsymbol{z}, 0) = E_0(\boldsymbol{z}) \end{cases} \quad (9)$$

Pugachev function:

$$\begin{aligned} \Phi(\boldsymbol{z}|\boldsymbol{y}, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\mathbb{E}\left[e^{i\boldsymbol{z}^T\Delta\boldsymbol{X}(t)} - 1|\boldsymbol{X}(t) = \boldsymbol{y}\right] = \\ = i\boldsymbol{z}^T\boldsymbol{a}(t, \boldsymbol{y}) - \frac{1}{2}\boldsymbol{z}^T\mathbf{H}(t, \boldsymbol{y})\mathbf{H}^T(t, \boldsymbol{y})\boldsymbol{z} \end{aligned} \quad (10)$$

Pugachev–Sveshnikov equation for SBM with DF

- Zayats, Sveshnikov (relay-type systems)
- Berezin, Zayats (2015)

Singular integral differential equation

$$\begin{cases} \dfrac{\partial E}{\partial t} + (z_1^2 - iz_2/2)E + (2\mu z_1 - z_2/2)\hat{E} - 2i\eta z_1 \Psi_0 = 0 \\ E(z_1, z_2; 0) = 1 \end{cases} \tag{11}$$

Hilbert transform:

$$\hat{E} = \frac{1}{\pi}\text{v.p.}\int_{-\infty}^{+\infty} \frac{E|_{z_1=s}}{s - z_1}\, ds \tag{12}$$

Additional integral term:

$$\Psi_0 = \frac{1}{2\pi}\text{v.p.}\int_{-\infty}^{+\infty} E|_{z_1=s}\, ds \tag{13}$$

The idea is to use the identity $\mathbb{E}\left[\operatorname{sign}(X(t))\ e^{izX(t)}\right] = -i\hat{E}$

Solution

- The method is similar to Carleman–Vekua regularization (see Gakhov "Boundary value problems")
  - First, solve equation (11) as if Ψ_0 is known
  - Then solve the Fredholm type equation of second kind for Ψ_0
  - In our case, an algebraic equation for Ψ_0
- Argument z_2 is considered as a parameter (fixed)

Cauchy-type integral (piecewise analytic function):

$$\Phi(\zeta, z_2; t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{E|_{z_1=s}}{s-\zeta} \, ds, \ \mathrm{Im}\ \zeta \neq 0 \qquad (14)$$

Half planes:

$$\Omega^+ = \{z \in \mathbb{C} |\ \mathrm{Im}\ z > 0\},\ \Omega^- = \{z \in \mathbb{C} |\ \mathrm{Im}\ z < 0\} \qquad (15)$$

$\Phi(\cdot, z_2; t)$ is analytic in $\Omega^+ \cup \Omega^-$

Boundary values of the Cauchy-type integral on the real line:

$$\Phi^{\pm}(z, z_2; t) = \lim_{\zeta \to z \pm i0} \Phi(\zeta, z_2; t), \quad \mathrm{Im}\ z = 0 \tag{16}$$

Sokhotski–Plemelj formulas:

$$\Phi^{+} - \Phi^{-} = E, \ \ \Phi^{+} + \Phi^{-} = -i\hat{E} \tag{17}$$

- Apply Laplace transform w.r.t. t and rewrite (11) thru $\tilde{\Phi}^{\pm} = \mathcal{L}(\Phi^{\pm})$

Riemann boundary value problem:

$$\begin{aligned}(z_1^2 + 2\mu i z_1 + p - iz_2)\tilde{\Phi}^{+} - i\eta z_1 \tilde{\Psi}_0 - \frac{1}{2} = \\ = (z_1^2 - 2\mu i z_1 + p)\tilde{\Phi}^{-} + i\eta z_1 \tilde{\Psi}_0 + \frac{1}{2}\end{aligned} \tag{18}$$

- $\Phi^{\pm}(z, z_2; t) = O(\frac{1}{|z|})$ as $z \to \infty$ such that $\mathrm{Im}\ z \gtrless 0$
- By Liouville's theorem:

$$\tilde{\Phi}^{\pm}(z_1, z_2; p) = \frac{\tilde{G}_0(z_2, p) + z_1 \tilde{G}_1(z_2, p) \pm i\eta z_1 \tilde{\Psi}_0(z_2; p) \pm 1/2}{z_1^2 \pm 2i\mu z_1 + p - (1 \pm 1)iz_2/2} \tag{19}$$

- Integrate (19) w.r.t. z_1, use the first Sokhotski–Plemelj formula and recall the definition of Ψ_0, then we get $\tilde{\Psi}_0 = -i\tilde{G}_1$
- Left-hand side of (19) can be analytically continued onto $\Omega^\pm$
- Its denominator has simple poles $i\nu^\pm = i(-\mu \pm \sqrt{\mu^2 + p - iz_2})$ and $i\varkappa^\pm = i(\mu \pm \sqrt{\mu^2 + p})$
- Singularities at $i\nu^+ \in \Omega^+$ and $i\varkappa^- \in \Omega^-$ should be removable

System of linear equations for $\tilde{G}_0$ and $\tilde{G}_1$:

$$G_0 + i\nu^+(1+\eta)G_1 + 1/2 = 0, \;\; G_0 + i\varkappa^-(1-\eta)G_1 - 1/2 = 0 \quad (20)$$

Characteristic functions:

$$\begin{aligned} \tilde{E}_X(z,p) &= \tilde{E}(z,0;p) = \frac{1}{2i\varkappa^-}\left(\frac{1+\eta}{z+i\varkappa^+} - \frac{1-\eta}{z-i\varkappa^+}\right) \\ \tilde{E}_\mathcal{I}(z,p) &= \tilde{E}(0,z;p) = \frac{(1+\eta)\varkappa^+ - (1-\eta)\nu^-}{\nu^-\varkappa^+((1-\eta)\varkappa^- - (1+\eta)\nu^+)} \end{aligned} \quad (21)$$

Theorem

The PDF of $X(t)$, the steady-state PDF of $X(t)$, and the PDF of the positive half-line occupation time $\mathcal{I}(t)$ have the following form:

$$f_X(x,t) = \left(\frac{1}{\sqrt{\pi t}} e^{-\frac{(|x|+2\mu t)^2}{4t}} + \mu e^{-2\mu|x|} \operatorname{Erfc}\left(\frac{|x|-2\mu t}{2\sqrt{t}} \right) \right) \cdot \begin{cases} \alpha, & x > 0, \\ 1-\alpha, & x < 0, \end{cases} \quad (22)$$

$$f_X^\infty(x) = f_X(x,+\infty) = 2\mu e^{-2\mu|x|}(\alpha \mathbb{1}_{(0,+\infty)}(x) + (1-\alpha)\mathbb{1}_{(-\infty,0]}(x)), \quad (23)$$

$$f_{\mathcal{I}}(y,t) = \frac{4e^{-\mu^2 t}}{\pi\sqrt{y(t-y)}} \int\limits_0^{+\infty} \int\limits_0^{+\infty} \chi(2\sqrt{y}s_1, 2\sqrt{t-y}s_2)\, s_1 s_2 e^{-s_1^2 - s_2^2}\, ds_1 ds_2, \quad (24)$$

where $0 < y < t,\ \alpha = (1+\eta)/2$, *and*

$$\chi(s_1,s_2) = \frac{1-\eta}{1+\eta} e^{-\mu\left(s_1 + \frac{\eta-3}{\eta+1}s_2\right)} \chi^+(s_1,s_2) + \frac{1+\eta}{1-\eta} e^{-\mu\left(\frac{\eta+3}{\eta-1}s_1 + s_2\right)} \chi^-(s_1,s_2),$$
$$\chi^+(s_1,s_2) = \mathbb{1}_{(0,+\infty)}((1+\eta)s_1 - (1-\eta)s_2), \quad \chi^-(s_1,s_2) = 1 - \chi^+(s_1,s_2). \quad (25)$$

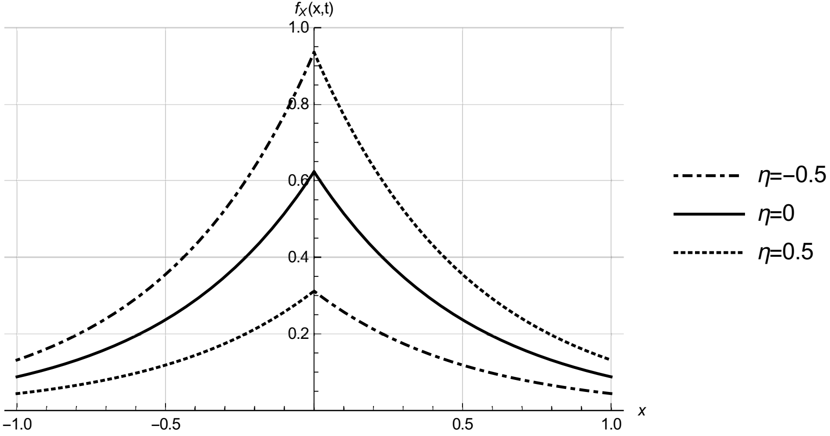


Figure: Probability density function $f_X(x,t)$ for different η ($\mu = 1, t = 1$).

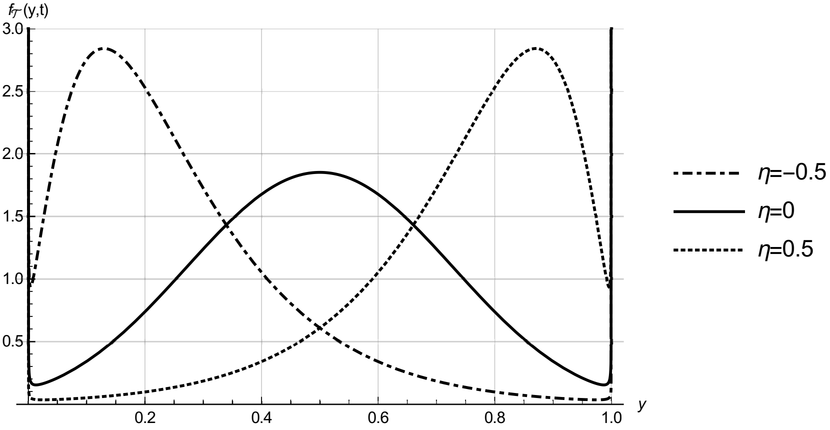


Figure: Scaled occupation time $\mathcal{T}(t) = \mathcal{I}(t)/t$ density function $f_{\mathcal{T}}(y,t)$ for different η ($\mu = 1, t = 2$).