Coupling and Convergence in Density and in Distribution

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Convergence in Distribution

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Let $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X})$ denote a coupling of $X_1, X_2, \ldots, X$.
This means that the random elements $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}$

(i) are all defined on the same probability space

(ii) and have the distributions $P_1, P_2, \ldots, P$.

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Let E be metric and $\mathcal{E}$ its Borel subsets. Let

$$X_n \to X \text{ in distribution as } n \to \infty$$

denote that for each bounded continuous $h : E \to \mathbb{R}$

$$\int h\, dP_n \to \int h\, dP \text{ as } n \to \infty.$$

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Theorem (Skorohod 1956)

Let E be separable and complete. Then

$$X_n \to X \text{ in distribution as } n \to \infty$$

$\iff$ $\exists$ a coupling $(\hat{X}_1, \hat{X}_2, \dots, \hat{X})$ of $X_1, X_2, \dots, X$ such that

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Theorem (Skorohod 1956 + Dudley 1968)

Let E be separable. Then

$$X_n \to X \text{ in distribution as } n \to \infty$$

$\Longleftrightarrow$ $\exists$ a coupling $(\hat{X}_1, \hat{X}_2, \dots, \hat{X})$ of $X_1, X_2, \dots, X$ such that

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Theorem (Skorohod 1956 + Dudley 1968 + Wichura 1970)

Let P have a separable support. Then

$$X_n \to X \text{ in distribution as } n \to \infty$$

$\Longleftrightarrow$ $\exists$ a coupling $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X})$ of $X_1, X_2, \ldots, X$ such that

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CONVERGENCE IN DENSITY

Let $X_1, X_2, \ldots, X$ be random elements in some space $(E, \mathcal{E})$ with distributions $P_1, P_2, \ldots, P$.

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(ii) and have the distributions $P_1, P_2, \ldots, P$.

Let $f_1, f_2, \ldots, f$ be densities of $X_1, X_2, \ldots, X$ w.r.t. some λ. Let

$X_n \to X$ in density as $n \to \infty$

denote that

$\liminf_{n\to\infty} f_n = f$ a.e. λ as $n \to \infty$.

Let $X_1, X_2, \ldots, X$ be random elements in some space $(E, \mathcal{E})$ with distributions $P_1, P_2, \ldots, P$.

Let $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X})$ denote a coupling of $X_1, X_2, \ldots, X$.
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$$X_n \to X \text{ in density as } n \to \infty$$

denote that

$$\liminf_{n\to\infty} f_n = f \quad \text{a.e. } \lambda \text{ as } n \to \infty.$$

Theorem (Thorisson 1995)

Let $(E, \mathcal{E})$ be general. Then

$$X_n \to X \text{ in density as } n \to \infty$$

$\iff \exists$ a coupling $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X})$ of $X_1, X_2, \ldots, X$ such that

$$\hat{X}_n \to \hat{X} \text{ pointwise in the discrete metric as } n \to \infty$$

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We shall prove this theorem in two steps:

Step 1: $\liminf_{n\to\infty} f_n$ is a density of X

$\Leftarrow$ $\exists$ a coupling such that $\hat{X}_n = \hat{X}, \ n \geq N$.

Step 2: $\liminf_{n\to\infty} f_n$ is a density of X

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$\Leftarrow$ $\exists$ a coupling such that $\hat{X}_n = \hat{X},\ n \geq N$.

Fix $B \in \mathcal{E}$ and $n < m$. Partition E into $A_n, \dots, A_m \in \mathcal{E}$ such that $\min_{n \leq i \leq m} f_i = f_j$ on A_j for $n \leq j \leq m$. Then

$$\mathbb{P}(\hat{X} \in B, N \leq n) = \sum_{j=n}^{m} \mathbb{P}(\hat{X} \in B \cap A_j, N \leq n) \quad \text{[partition]}$$

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$$= \sum_{j=n}^{m} \mathbb{P}(\hat{X}_j \in B \cap A_j, N \leq n) \quad \text{[since } \hat{X}_j = \hat{X} \text{ when } j \geq n \geq N\text{]}$$

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$$\leq \sum_{j=n}^{m} \mathbb{P}(\hat{X}_j \in B \cap A_j) = \sum_{j=n}^{m} \int_{B \cap A_j} f_j = \int_B \min_{n \leq i \leq m} f_i \leq 1\,.$$

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Send first $m \to \infty$ and then $n \to \infty$ to obtain

$$P(B) \leq \int_B \liminf_{n\to\infty} f_n \leq 1 \quad \text{for all } B \in \mathcal{E}.$$

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Send first $m \to \infty$ and then $n \to \infty$ to obtain

$$P(B) \leq \int_B \liminf_{n\to\infty} f_n \leq 1 \quad \text{for all } B \in \mathcal{E}.$$

This forces $P(B) = \int_B \liminf_{n\to\infty} f_n$ for all $B \in \mathcal{E}$.

Step 2: $\liminf_{n\to\infty} f_n$ is a density of X

$\Rightarrow$ $\exists$ a coupling such that $\hat{X}_n = \hat{X},\ n \geq N$.

Let μ_n have density $g_n := \inf_{m\geq n} f_m$. Then $\mu_1 \leq \mu_2 \leq \cdots \nearrow P$.

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Let $N, V_1, V_2, \ldots, W_1, W_2, \ldots$ be independent random elements.

Let N have distribution function $\mathbb{P}(N \leq n) = \mu_n(E), \quad n \in \mathbb{N}.$

Let V_n have distribution $(\mu_n - \mu_{n-1})/\mathbb{P}(N = n)$ where $\mu_0 = 0$.

Let W_n have distribution $(P_n - \mu_n)/\mathbb{P}(N > n)$.

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Put $\hat{X} = V_N$. Then

$$\mathbb{P}(\hat{X} \in \cdot) = \sum_i \mathbb{P}(V_i \in \cdot)\mathbb{P}(N = i) = \sum_i (\mu_i - \mu_{i-1}) = P.$$

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$$\mathbb{P}(\hat{X} \in \cdot) = \textstyle\sum_i \mathbb{P}(V_i \in \cdot)\mathbb{P}(N = i) = \sum_i (\mu_i - \mu_{i-1}) = P.$$

Put $\hat{X}_n = V_N$ on $\{N \leq n\}$ and $\hat{X}_n = W_n$ on $\{N > n\}$. Then

$$\mathbb{P}(\hat{X}_n \in \cdot) = \sum_{1\leq i\leq n} \mathbb{P}(V_i \in \cdot)\mathbb{P}(N = i) + \mathbb{P}(W_n \in \cdot)\mathbb{P}(N > n)$$

$$= \sum_{1\leq i\leq n} (\mu_i - \mu_{i-1}) + (P_n - \mu_n) = P_n\,.$$

Clearly $\hat{X}_n = \hat{X},\ n \geq N$.

We just proved the following result:

Theorem (Thorisson 1995)

Let $(E, \mathcal{E})$ be general. Then

$$X_n \to X \quad \text{in density as} \quad n \to \infty$$

$\iff$ $\exists$ a coupling $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X})$ of $X_1, X_2, \ldots, X$ and an $\mathbb{N}$-valued N such that

$$\hat{X}_n = \hat{X}, \quad n \geq N.$$

This result can be extended to stochastic processes considered in widening time windows.

We shall only present the discrete-time, discrete-space case but the result holds for continous-time Skorohod space.

Set $\mathbf{Z} = (Z^1, Z^2, \dots, X)$ and $\mathbf{Z}_n = (Z_n^1, Z_n^2, \dots, X_n)$.

For a time window of length $k \in \mathbb{N}$ write

$$\mathbf{Z}^k = (Z^1, Z^2, \dots, Z^k) \quad \text{and} \quad \mathbf{Z}_n^k = (Z_n^1, Z_n^2, \dots, Z_n^k).$$

Theorem (Thorisson 2016)

If Z^i, Z_n^i are $\mathbb{N}$-valued and X, X_n are general then

$$\forall k \in \mathbb{N}, \mathbf{i}^k \in \mathbb{N}^k: \quad \mathbb{P}(\mathbf{Z}_n^k = \mathbf{i}^k) \to \mathbb{P}(\mathbf{Z}^k = \mathbf{i}^k) \text{ as } n \to \infty \qquad (*)$$

$\iff$ $\exists$ a coupling $(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_2, \dots, \hat{\mathbf{Z}})$ of $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}$, an $\mathbb{N}$-valued N and $0 \le k_1 \le k_2 \le \cdots \to \infty$ such that

$$\hat{\mathbf{Z}}_n^{k_n} = \hat{\mathbf{Z}}^{k_n}, \quad n \ge N.$$

Note: Since $\mathbb{N}$ is discrete, $(*)$ is equivalent to

$$\forall k \in \mathbb{N}: \quad \mathbf{Z}_n^k \to \mathbf{Z}^k \text{ in density as } n \to \infty$$

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This theorem yields the Skorohod Representation Theorem.

Skorohod Representation Theorem

Let $P(S) = 1$ where $S \in \mathcal{E}$ is separable. If

$$P_n(A) \to P(A) \text{ for } P\text{-continuity } A \in \mathcal{E}, \quad n \to \infty,$$

then $\exists$ a coupling $(\hat{X}_1, \hat{X}_2, \dots, \hat{X})$ of $X_1, X_2, \dots, X$ such that

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Proof: Let $A_2, A_3, \ldots$ be disjoint S-covering P-continuity sets of diameter < 1. Then $A_1 = E \setminus (A_2 \cup A_3 \cup \ldots)$ is P-continuity set.

Put $A_{11} = A_1$ and $A_{12} = A_{13} = \cdots = \emptyset$. For $i > 1$, let $A_{i2}, A_{i3}, \ldots$ be disjoint P-continuity sets of diameter $< 1/2$ covering $S \cap A_i$. Then $A_{i1} = A_i \setminus (A_{i2} \cup A_{i3} \cup \ldots)$ is a P-continuity set.

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Continue this recursively to obtain that:

$\{A_{\mathbf{i}^k} : \mathbf{i}^k \in \mathbb{N}^k\}$ are nested P-continuity E-partitions such that $A_{\mathbf{i}^k}, \mathbf{i}^k \in (\mathbb{N} \setminus \{1\})^k$, cover S and are each of diameter $< 1/k$. $(*)$

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Put $\quad \mathbf{Z}_n = (Z_n^1, Z_n^2, \ldots, X_n) \quad$ and $\quad \mathbf{Z} = (Z^1, Z^2, \ldots, X)$

where Z_n^k and Z^k are defined as follows: for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\mathbf{Z}_n^k = \mathbf{i}^k \text{ if } X_n \in A_{\mathbf{i}^k} \qquad \text{and} \qquad \mathbf{Z}^k = \mathbf{i}^k \text{ if } X \in A_{\mathbf{i}^k}.$$

Now $P_n(A_{\mathbf{i}^k}) \to P(A_{\mathbf{i}^k})$ yields $\mathbb{P}(\mathbf{Z}_n^k = \mathbf{i}^k) \to \mathbb{P}(\mathbf{Z}^k = \mathbf{i}^k)$ as $n \to \infty$

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Put $\mathbf{Z}_n = (Z_n^1, Z_n^2, \ldots, X_n)$ and $\mathbf{Z} = (Z^1, Z^2, \ldots, X)$
where Z_n^k and Z^k are defined as follows: for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\mathbf{Z}_n^k = \mathbf{i}^k \text{ if } X_n \in A_{\mathbf{i}^k} \quad \text{and} \quad \mathbf{Z}^k = \mathbf{i}^k \text{ if } X \in A_{\mathbf{i}^k}.$$

Now $P_n(A_{\mathbf{i}^k}) \to P(A_{\mathbf{i}^k})$ yields $\mathbb{P}(\mathbf{Z}_n^k = \mathbf{i}^k) \to \mathbb{P}(\mathbf{Z}^k = \mathbf{i}^k)$ as $n \to \infty$

Apply the **corollary** to obtain a coupling $(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_2, \ldots, \hat{\mathbf{Z}})$ and an $\mathbb{N}$-valued random variable N such that $\hat{\mathbf{Z}}_n^{k_n} = \hat{\mathbf{Z}}^{k_n}$, $n \geq N$. $(**)$

Also $\hat{X} \in S$ and for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\hat{\mathbf{Z}}_n^k = \mathbf{i}^k \text{ if } \hat{X}_n \in A_{\mathbf{i}^k} \quad \text{and} \quad \hat{\mathbf{Z}}^k = \mathbf{i}^k \text{ if } \hat{X} \in A_{\mathbf{i}^k}.$$

$\{A_{\mathbf{i}^k} : \mathbf{i}^k \in \mathbb{N}^k\}$ are nested P-continuity E-partitions such that $A_{\mathbf{i}^k}, \mathbf{i}^k \in (\mathbb{N} \setminus \{1\})^k$, cover S and are each of diameter $< 1/k$. $(*)$

Put $\mathbf{Z}_n = (Z_n^1, Z_n^2, \ldots, X_n)$ and $\mathbf{Z} = (Z^1, Z^2, \ldots, X)$

where Z_n^k and Z^k are defined as follows: for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

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Now $P_n(A_{\mathbf{i}^k}) \to P(A_{\mathbf{i}^k})$ yields $\mathbb{P}(\mathbf{Z}_n^k = \mathbf{i}^k) \to \mathbb{P}(\mathbf{Z}^k = \mathbf{i}^k)$ as $n \to \infty$

Apply the **corollary** to obtain a coupling $(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_2, \ldots, \hat{\mathbf{Z}})$ and an $\mathbb{N}$-valued random variable N such that $\hat{\mathbf{Z}}_n^{k_n} = \hat{\mathbf{Z}}^{k_n}$, $n \geq N$. $(**)$

Also $\hat{X} \in S$ and for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\hat{\mathbf{Z}}_n^k = \mathbf{i}^k \text{ if } \hat{X}_n \in A_{\mathbf{i}^k} \quad \text{and} \quad \hat{\mathbf{Z}}^k = \mathbf{i}^k \text{ if } \hat{X} \in A_{\mathbf{i}^k}.$$

Thus $\hat{X}_n \in A_{\hat{\mathbf{Z}}_n^{k_n}}$ and $\hat{X} \in A_{\hat{\mathbf{Z}}^{k_n}}$ for all n. Apply $(**)$ to obtain:

for $n \geq N$: both $\hat{X}_n$ and $\hat{X} \in A_{\hat{\mathbf{Z}}^{k_n}}$.

$\{A_{\mathbf{i}^k} : \mathbf{i}^k \in \mathbb{N}^k\}$ are nested P-continuity E-partitions such that $A_{\mathbf{i}^k}, \mathbf{i}^k \in (\mathbb{N} \setminus \{1\})^k$, cover S and are each of diameter $< 1/k$. $(*)$

Put $\mathbf{Z}_n = (Z_n^1, Z_n^2, \ldots, X_n)$ and $\mathbf{Z} = (Z^1, Z^2, \ldots, X)$ where Z_n^k and Z^k are defined as follows: for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\mathbf{Z}_n^k = \mathbf{i}^k \text{ if } X_n \in A_{\mathbf{i}^k} \quad \text{and} \quad \mathbf{Z}^k = \mathbf{i}^k \text{ if } X \in A_{\mathbf{i}^k}.$$

Now $P_n(A_{\mathbf{i}^k}) \to P(A_{\mathbf{i}^k})$ yields $\mathbb{P}(\mathbf{Z}_n^k = \mathbf{i}^k) \to \mathbb{P}(\mathbf{Z}^k = \mathbf{i}^k)$ as $n \to \infty$

Apply the **corollary** to obtain a coupling $(\hat{\mathbf{Z}}_1, \hat{\mathbf{Z}}_2, \ldots, \hat{\mathbf{Z}})$ and an $\mathbb{N}$-valued random variable N such that $\hat{\mathbf{Z}}_n^{k_n} = \hat{\mathbf{Z}}^{k_n}$, $n \geq N$. $(**)$

Also $\hat{X} \in S$ and for $k \in \mathbb{N}$, $\mathbf{i}^k \in \mathbb{N}^k$,

$$\hat{\mathbf{Z}}_n^k = \mathbf{i}^k \text{ if } \hat{X}_n \in A_{\mathbf{i}^k} \quad \text{and} \quad \hat{\mathbf{Z}}^k = \mathbf{i}^k \text{ if } \hat{X} \in A_{\mathbf{i}^k}.$$

Thus $\hat{X}_n \in A_{\hat{\mathbf{Z}}_n^{k_n}}$ and $\hat{X} \in A_{\hat{\mathbf{Z}}^{k_n}}$ for all n. Apply $(**)$ to obtain:

for $n \geq N$: both $\hat{X}_n$ and $\hat{X} \in A_{\hat{\mathbf{Z}}^{k_n}}$.

Now apply $(*)$: Since $\hat{X} \in S$, we have $\hat{\mathbf{Z}}^{k_n} \in (\mathbb{N} \setminus \{1\})^k$ and thus

for $n \geq N$: $d(\hat{X}_n, \hat{X}) < 1/k_n$. And $1/k_n \to 0$ as $n \to \infty$.

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