Sensitivity analysis of renewable reliability systems.

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ABSTRACT

- The problem of reliability characteristics sensitivity to the shapes of some input distributions is considered.
- A short review of the latest investigations on this direction is proposed.
- More detailed the heterogeneous double redundant hot standby renewable reliability system is considered.
- Time dependent, stationary and quasi-stationary characteristics for such systems are calculated.

Outline

Introduction and Motivation

- Stability of different systems characteristics to the changes in initial states or exterior factors are the **key problems** in all natural sciences. For stochastic systems stability often means **insensitivity** or **low sensitivity** of their output characteristics to the shapes o f some input distributions.
- B. Sevast'yanov (1957) [1] shown the insensitivity of the lost probability in Erlang's formulas on the shape of service time.
- I.Kovalenko (1976) in [2] shown that the necessary and sufficient conditions for insensitivity of stationary reliability characteristics of redundant renewable system with exponential life time and general repair time distributions of its components to the shape of the latter consist in sufficient amount of repairing facilities.
- The sufficiency of this condition for the case of general life and repair time distributions has been found by V.Rykov (2013) in [3]. However, in the case of limited possibilities for restoration these results do not hold [4].

- In series of work of B.V. Gnedenko (1964), A.D. Solov'ev (1970) [5, 6, 7] it was shown that under "quick" restoration the reliability function of a cold standby double redundant heterogeneous system tends to the exponential one for any life and repair time distributions of its elements.
- This result also means the asymptotical insensitivity of the reliability characteristics of such system to the shapes of their elements life and repair times distributions.
- In the papers [8, 9, 10] the problem of systems' steady state reliability characteristics sensitivity to the shape of life and repair time distributions of their components for the same type of systems has been considered, for the case, when one of the input distributions (either of life or repair time lengths) is exponential.
- For these models explicit expressions for stationary probabilities have been obtained which show their evident dependence on the non-exponential distributions in the form of their Laplace-Stiltjes transforms.

- The problem of the convergence rate in the paper of V.V. Kalashnikov (1997) [14] has been considered, where the evaluation of the convergence rate has been done in terms of moments of appropriate distributions.
- The numerical investigation and simulation results, given in [11, 12, 13] demonstrate enough quick appearance of practical insensitivity of the time dependent as well as stationary reliability characteristics to the shapes of life and repair time distributions with fixed their mean values.
- In this presentation the previous results review will be done and they will be extended for the case of heterogeneous double redundant standby renewable systems.
- The talk ends with conclusion and some problems description.

The Problem setting and notations

Consider a heterogeneous hot double redundant repairable reliability system.
СЮДА БЫ РИСУНОЧЕК !!!!!

- Life times of components are exponentially distributed r.v. with parameters α_1 and α_2.
- The repair times of components have absolute continuous distributions with c.d.f. $B_k(x)$ $(k = 1, 2)$ and p.d.f. $b_k(x)$ $(k = 1, 2)$.
- All life and repair times are independent.
- The "up" (working) states of each component will be marked by 0 and the "down" (failed) state by 1.

Under considered assumptions the system state space can be represented as $E = \{0, 1, 2, 3\}$, which means:

- 0 — both components are working,
- 1 — the first component is repaired, and the second one is working,
- 2 — the second component is repaired, and the first one is working,
- 3 — both components are in down states, system is failed and repaired.

For the system behavior description introduce a random process $J = \{J(t),\ t \geq 0\}$ with values into system set of states E:

$$J(t) = i, \quad \text{if in the time } t \text{ the system is in the state } i \in E.$$

At that the system states subset $E_0 = \{0, 1, 2\}$ represents its working (up) states of the system, and the subset $E_1 = \{3\}$ represents the system failure (down) state. Denote also by

- $\alpha = \alpha_1 + \alpha_2$ the summary intensity of the system failure;
- $b_k = \int_0^\infty (1 - B_k(x))dx$ k-th element repair time expectations;
- $\beta_k(x) = (1 - B_k(x))^{-1} b_k(x)$ k-th element conditional repair intensity given elapsed repair time is x;
- $\tilde{b}_k(s) = \int_0^\infty e^{-sx} b_k(x)dx$ Laplace transform (LT) of the k-th element repair time distribution.
- $T = \inf\{t : J(t) \in E_1\}$ the system life time

In this paper we are interesting in study of

- the **reliability function**

$$R(t) = \mathbf{P}\{T > t\}$$

  or system **life time distribution** $F(t) = 1 - R(t)$;

- the system **steady-state probabilities** (s.s.p.)

$$\pi_j = \lim_{t\to\infty} \mathbf{P}\{J(t) = j\};$$

- however because any system does not exist infinitely long for practice is more interesting characteristic is so called **quasi stationary probabilities** (q.s.p.)

$$\bar{\pi}_j = \lim_{t\to\infty} \mathbf{P}\{J(t) = j | t \le T\}.$$

The reliability function calculation

For the system investigation the Markovization method is used. To realize it consider the two-dimensional Markov process $Z = \{Z(t),\ t \geq 0)\}$, with $Z(t) = (J(t), X(t))$ where

- $J(t)$ represents the system state, and
- $X(t)$ is an additional variable, which means the elapsed repair time of $J(t)$-th component at time t.

The process phase space equals to $\mathcal{E} = \{0, (1, x), (2, x), 3\}$, which mean:

- 0 – both components are working,
- (i, x) – the i-th component is failed and repairing, and its elapsed repair time equal to x, while another one is working,
- 3 – both components are failed, and therefore the system is failed.

Appropriate probabilities are denoted by

$$\pi_0(t),\ \pi_1(t;x),\ \pi_2(t;x),\ \pi_3(t).$$

The state transition graph of the system is represented in figure 1.

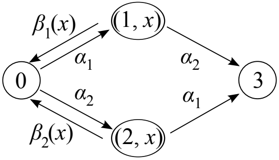

Figure 1: Absorbing system transition graph

By usual method the following Kolmogorov forward system of partial differential equations for these probabilities can be obtained,

$$\begin{aligned}
&\frac{d}{dt}\pi_0(t) = -\alpha\pi_0(t) + \int_0^t \pi_1(t,u)\beta_1(u)du + \int_0^t \pi_2(t,u)\beta_1(u)du,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_1(t;x) = -(\alpha_2 + \beta_1(x))\pi_1(t;x),\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_2(t;x) = -(\alpha_1 + \beta_2(x))\pi_2(t;x),\\
&\frac{d}{dt}\pi_3(t) = \alpha_1\int_0^t \pi_2(t;u)du + \alpha_2\int_0^t \pi_1(t;u)du.
\end{aligned} \tag{1}$$

jointly with the initial $\pi_0(0) = 1$ and boundary conditions

$$\pi_1(t,0) = \alpha_1\pi_0(t), \quad \pi_2(t,0) = \alpha_2\pi_0(t). \tag{2}$$

Theorem

The LT $\tilde{\pi}_(s)$ and $\tilde{R}(s)$ of the $\pi_i(t)$ and the reliability function $R(t)$ are

$$\begin{aligned}
\tilde{\pi}_0(s) &= \frac{1}{s+\psi(s)}, \\
\tilde{\pi}_1(s) &= \alpha_1 \frac{1-\tilde{b}_1(s+\alpha_2)}{(s+\alpha_2)(s+\psi(s))}, \\
\tilde{\pi}_2(s) &= \alpha_2 \frac{1-\tilde{b}_2(s+\alpha_1)}{(s+\alpha_1)(s+\psi(s))}, \\
\tilde{\pi}_3(s) &= \frac{\alpha_1\alpha_2(\phi_1(s)+\phi_2(s)}{s(s+\alpha_1)(s+\alpha_2)(s+\psi(s))}, \\
\tilde{R}(s) &= \frac{(s+\alpha_1)(s+\alpha_2)+\alpha_1\phi_1(s)+\alpha_2\phi_2(s)}{(s+\alpha_1)(s+\alpha_2)(s+\psi(s))}, \qquad (3)
\end{aligned}$$

where the following notations are used

$$\begin{aligned}
\phi_i(s) &= (s+\alpha_i)(1-\tilde{b}_i(s+\alpha_{i^*})), \ (i=1,2), \\
\psi(s) &= \alpha_1(1-\tilde{b}_1(s+\alpha_2))+\alpha_2(1-\tilde{b}_2(s+\alpha_1)).
\end{aligned}$$

with $i^*=2$ for $i=1$, and $i^*=1$ for $i=2$.

Proof

The **proof** is not too complicated, but enough long. It uses the method of characteristic for partial differential equations solution and some calculations based on Laplace transform. It'll be done in the full paper for conference journal.

Stationary probabilities

For the system stationary regime study we need to determine the system behavior after its failure. There are at least two possibilities:

- **full repair**, when after the system failure the renewal of whole system begins that demand some random time with, say c.d.f. $B_3(t)$, and after this time the system goes to the state 0;
- **partial repair**, when after failure the system prolong to work in the same regime, i.e. the repaired element prolong to be repaired and after its renewal the system goes to the state 1 or 2 dependently on what type of component is repaired in state 3. Therefore we need to divide this state into two states $(3,1)$ which means that both elements fail and the first one is repaired, and $(3,2)$ which means that both elements fail and the second one is repaired.

Partial repair

In case of partial repair (note that the distributions to the first failure and between failures are different in this case) the transition graph represented in the picture 2.

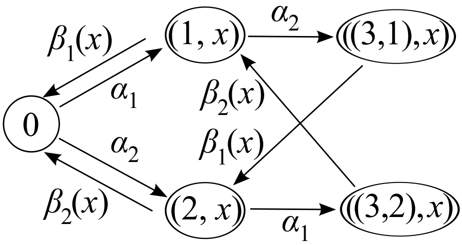

Figure 2: Transition graph of the system with an absorbing state

To calculate steady-state probabilities consider two-dimensional Markov process $Z = \{Z(t),\ t \geq 0)\}$, with $Z(t) = (J(t), X(t))$ where

- $J(t)$ represents the system state, and
- $X(t)$ is an additional variable, which means the elapsed time of the element under recovery at time t.

The process phase space equals to $\mathcal{E} = \{0, (1, x), (2, x), 3\}$, which mean:

- 0 – both components of system are working,
- (i, x) – the i-th component (i=1, 2) is failed and repaired, and its elapsed repair time equal to x, and the other one is working,
- $((3, i), x)$ – both elements are failed, and i-th one is repaired with elapsed time equal x.

Appropriate probabilities are denoted by

$$\pi_0(t),\ \pi_1(t; x),\ \pi_2(t; x),\ \pi_{(3,1)}(t; x),\ \pi_{(3,2)}(t; x).$$

The following Kolmogorov forward system of equations holds

$$\frac{d}{dt}\pi_0(t) = -\alpha\pi_0(t) + \int\limits_0^t \pi_1(t,u)\beta_1(u)du + \int\limits_0^t \pi_2(t,u)\beta_2(u)du,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_i(t;x) = -(\alpha_{i^*} + \beta_i(x))\pi_(t;x), \ (i = 1,2)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_{(3,i)}(t;x) = -\beta_i(x)\pi_{(3,i)}(t;x) + \alpha_{i^*}\pi_i(t;x), \ (i = 1,2) \quad (4)$$

with the initial condition $\pi_0(0) = 0$, and the boundary conditions of the form

$$\pi_1(t;0) = \alpha_1\pi_0(t) + \int\limits_0^t \pi_{(3,2)}(t;u)\beta_2(u)du,$$

$$\pi_2(t;0) = \alpha_2\pi_0(t) + \int\limits_0^t \pi_{(3,1)}(t;u)\beta_1(u)du. \quad (5)$$

Here and later $i^* = 2$ for $i = 1$ and vice verse.

Because the state 0 represents a positive atom for the process Z, the limiting probabilities exist

$$\pi_0 = \lim_{t\to\infty} \pi_0(t), \;\; \pi_i(x) = \lim_{t\to\infty} \pi_i(t;x), \quad (i = 1, 2, (3,1), (3,2)).$$

and satisfy to the system of balance equations

$$\begin{aligned} \alpha\pi_0 &= \int_0^\infty \pi_1(u)\beta_1(u)du + \int_0^\infty \pi_2(u)\beta_2(u)du, \\ \frac{d}{dx}\pi_i(x) &= -(\alpha_{i^*} + \beta_i(x))\pi_i(x), \;\; (i = 1, 2) \\ \frac{d}{dx}\pi_{(3,i)}(x) &= -\beta_i(x)\pi_{(3,i)}(x) + \alpha_{i^*}\pi_i(x), \;\; (i = 1, 2). \end{aligned} \tag{6}$$

with appropriate boundary conditions

$$\pi_i(0) = \alpha_1\pi_0 + \int_0^\infty \pi_{(3,i^*)}(u)\beta_{i^*}(u)du, \;\; (i = 1, 2) \tag{7}$$

Theorem

The s.s.p. of the system with partial repair has the form

$$\begin{aligned} \pi_i(x) &= C_i e^{-\alpha_{i^*} x}(1 - B_i(x)), \ (i = 1, 2) \\ \pi_{(3,i)}(x) &= C_i(1 - e^{-\alpha_{i^*} x})(1 - B_i(x)), \ (i = 1, 2) \end{aligned} \tag{8}$$

where

$$C_i = \frac{\Delta_i}{\Delta}\pi_0, \quad (i = 1, 2), \tag{9}$$

with

$$\begin{aligned} \Delta &= 1 - (1 - \tilde{b}_1(\alpha_2))(1 - \tilde{b}_2(\alpha_1)), \\ \Delta_i &= \alpha_i - \alpha_{i^*}(1 - \tilde{b}_{i^*}(\alpha_i)) \quad (i = 1, 2), \end{aligned}$$

and

$$\pi_0 = \frac{\Delta}{\Delta + \Delta_1 b_1 + \Delta_2 b_2} \tag{10}$$

Proof

The proof of the theorem uses the usual method of the variables division for the first two equations, and then the method of the constant variation for the two next.

Application of the boundary condition allows to get the result.

The detailed proof in the full paper will be done.

The above formulas demonstrate an **evident dependence** of the system s.s.p. of the shapes of elements repair time distributions.

Corollary

Macro-states s.s.p. are

$$
\begin{aligned}
\pi_1 &= C_1 \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2}, \\
\pi_2 &= C_2 \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1}, \\
\pi_{(3,1)} &= C_1 b_1 \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2}, \\
\pi_{(3,2)} &= C_2 b_2 \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1}. \qquad (11)
\end{aligned}
$$

with the same values of C_1, C_2, Δ_1, Δ_2, Δ and π_0.

Remark

With the help of normalizing condition one can get some simple connection between constants $C_1,\ C_2$. Really dividing representation for π_0 into sum of two summands and summing the expression for stationary probabilities one can get

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_{(3,1)} + \pi_{(3,2)} =$$

$$C_1\left(\frac{\tilde{b}_1(\alpha_2)}{\alpha_1} + \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2} + b_1 - \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2}\right) +$$

$$C_2\left(\frac{\tilde{b}_2(\alpha_1)}{\alpha_2} + \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1} + b_2 - \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1}\right)$$

$$C_1\left(\frac{\tilde{b}_1(\alpha_2)}{\alpha_2} + b_1\right) + C_2\left(\frac{\tilde{b}_2(\alpha_1)}{\alpha_1} + b_2\right). \qquad (12)$$

Corollary

Using last expression one can get appropriate formulas for s.s.p. of homogeneous system, which coincide with those obtained before, see for example [8, 10].

Denote for homogeneous system the macro-state probabilities by $\bar{\pi}_0, \bar{\pi}_1, \bar{\pi}_2$ and by $\rho = \alpha b$.

Corollary

The SSP for homogeneous system equals

$$\begin{aligned}
\bar{\pi}_0 &= \frac{\tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}, \\
\bar{\pi}_1 &= \frac{1 - \tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}, \\
\bar{\pi}_2 &= \frac{\rho + 1 - \tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}.
\end{aligned}$$

Proof

In homogeneous case the expression (12) with notation $\rho = \alpha b$ gives the following result

$$C_1 + C_2 = \frac{\alpha}{\tilde{b}(\alpha) + \rho}.$$

Thus by summing the macro-states probabilities with the above notations $\bar{\pi}_0, \bar{\pi}_1, \bar{\pi}_2$ and $\bar{\pi}_0 = \pi_0, \bar{\pi}_1 = \pi_1 + \pi_2, \bar{\pi}_2 = \pi_{(3,1)} + \pi_{(3,2)}$ it gives

$$\begin{aligned}
\bar{\pi}_0 &= (C_1 + C_2)\frac{\tilde{b}(\alpha)}{\alpha} = \frac{\tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}, \\
\bar{\pi}_1 &= \pi_1 + \pi_2 = (C_1 + C_2)\frac{1 - \tilde{b}_1(\alpha)}{\alpha} = \frac{1 - \tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}, \\
\bar{\pi}_2 &= \pi_{(3,1)} + \pi_{(3,2)} = (C_1 + C_2)\left(b - \frac{1 - \tilde{b}_1(\alpha)}{\alpha}\right) = \frac{\rho + 1 - \tilde{b}(\alpha))}{\rho + \tilde{b}(\alpha)}
\end{aligned}$$

that proof the Corollary.

Full repair

In the case of system full repair after its failure the system behavior can be described by the process $J = \{J(t),\ t \geq 0\}$ with the state space $E = \{0,\ 1,\ 2,\ 3\}$, which means:

- 0 — both components are in "up" states,
- i ($i = 1, 2$) — the i-th component is repaired and the other is working,
- 3 — both components are in down states, system is failed.

In this case the system pass from the state 3 to the state 0 with some absolutely continuous c.d.f, say $B_3(x)$, and p.d.f. $b_3(x)$ and transition intensity (conditional p.d.f. given elapsed summary system repair time equals to x) equals $\beta_3(x)$.

For the system investigation introduce the two-dimensional Markov process

$$Z = \{J(t),\ X(t),\ \ t \geq 0\}$$

under phase space

$$\mathcal{E} = \{0, 1, 2, 3\} \times R_+ = \{0, (1, x), (2, x), 3\},$$

which means:

- 0 – both elements are working,
- (i, x) $(i = 1, 2)$ – the i-th component is repaired and its elapsed repair time equal to x, the other one is working,
- $(3, x)$ – both elements are failed, and the system elapsed repair time equal to x.

Appropriate probabilities are denoted by

$$\pi_0(t),\ \pi_1(t; x),\ \pi_2(t; x),\ \pi_3(t; x).$$

The transition graph of the process represented at the picture below.

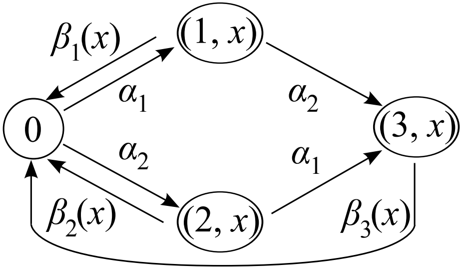

Figure 3: Transition graph of the system with an absorbing state

The following Kolmogorov forward system of differential equations holds

$$
\begin{aligned}
\frac{d}{dt}\pi_0(t) &= -\alpha\pi_0(t) + \int_0^t \pi_1(t,u)\beta_1(u)du + \int_0^t \pi_2(t,u)\beta_2(u)du + \\
&+ \int\limits_0^t \beta_3(u)\pi_3(u)du, \\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_i(t;x) = -(\alpha_{i^*} + \beta_i(x))\pi_i(t;x), \ \ (i = 1,2) \\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_3(t;x) = -\beta_3\pi_3(t;x). \qquad (13)
\end{aligned}
$$

with the initial $\pi_0(t) = \delta(t)$ and boundary conditions of the form

$$
\begin{aligned}
\pi_i(t,0) &= \alpha_i\pi_0(t), \ \ (i = 1,2) \\
\pi_3(t,0) &= \alpha_1 \int\limits_0^t \pi_2(t,u)du + \alpha_2 \int\limits_0^t \pi_1(t,u)du. \qquad (14)
\end{aligned}
$$

The process Z is a positive recurrent one (the state 0 is its positive atom) and therefore it has the limiting under $t \to \infty$ probabilities, which coincide with s.s.p.

$$\pi_0 = \lim_{t\to\infty} \pi_0(t), \quad \pi_i(x) = \lim_{t\to\infty} \pi_i(t;x) \ \ (i \in \{1,2,3\})$$

for which the system of balance equations holds

$$\begin{aligned}
\alpha\pi_0 &= \int_0^t \pi_1(t,u)\beta_1(u)du + \int_0^t \pi_2(t,u)\beta_1(u)du + \int_0^t \beta_3(u)\pi_3(u)du, \\
\frac{d}{dx}\pi_i(x) &= -(\alpha_{i^*} + \beta_i(x))\pi_i(x), \ \ (i = 1,2) \\
\frac{d}{dx}\pi_3(x) &= -\beta_3(x)\pi_3(t;x) \qquad (15)
\end{aligned}$$

with the boundary conditions of the form

$$\begin{aligned}
\pi_i(0) &= \alpha_i\pi_0, \ \ (i = 1,2), \\
\pi_3(0) &= \alpha_1 \int_0^\infty \pi_2(u)du + \alpha_2 \int_0^\infty \pi_1(t,u)du. \qquad (16)
\end{aligned}$$

Theorem

The s.s.p. for the system under full repair has the form

$$\begin{aligned} \pi_i(x) &= \alpha_i e^{-\alpha_{i*}x}(1 - B_1(x))\pi_0, \quad (i = 1, 2) \\ \pi_3(x) &= [\alpha_1(1 - \tilde{b}_1(\alpha_2)) + \alpha_2(1 - \tilde{b}_1(\alpha_2))](1 - B_3(x))\pi_0. \end{aligned} \tag{17}$$

where π_0 is given by

$$\pi_0 = \left[1 + (1 - \tilde{b}_1(\alpha_2))\left(\frac{\alpha_1}{\alpha_2} + \alpha_1 b_3\right) + (1 - \tilde{b}_2(\alpha_1))\left(\frac{\alpha_2}{\alpha_1} + \alpha_2 b_3\right)\right]^{-1}, \tag{18}$$

with $b_3 = \mathbf{E}[B_3] = \int_0^\infty (1 - B_3(x))dx$,

Corollary

Appropriate stationary macro-state s.s.p. are

$$
\begin{aligned}
\pi_1 &= \frac{\alpha_1}{\alpha_2}(1 - \tilde{b}_1(\alpha_2))\pi_0, \\
\pi_2 &= \frac{\alpha_2}{\alpha_1}(1 - \tilde{b}_2(\alpha_1))\pi_0, \\
\pi_3 &= [\alpha_1(1 - \tilde{b}_1(\alpha_2)) + \alpha_2(1 - \tilde{b}_1(\alpha_2))]b_3\pi_0,
\end{aligned}
$$

with the same value of π_0.

Proof

Solutions of the last three equations of the system (15) are

$$\begin{aligned}\pi_1(x) &= C_1 e^{-\alpha_2 x}(1 - B_1(x)),\\ \pi_2(x) &= C_2 e^{-\alpha_1 x}(1 - B_2(x)),\\ \pi_3(x) &= C_3(1 - B_3(x)).\end{aligned}$$

Using boundary conditions (16) to find unknown constants C_i gives

$$C_1 = \alpha_1\pi_0, \;\; C_2 = \alpha_2\pi_0, \;\; C_3 = [\alpha_1(1 - \tilde{b}_1(\alpha_2)) + \alpha_2(1 - \tilde{b}_2(\alpha_1))]\pi_0.$$

Using normalizing conditions in order to find probability π_0 gives

$$\begin{aligned}1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 =\\ &= \left[1 + \alpha_1 \frac{1 - \tilde{b}_1(\alpha_2)}{\alpha_2} + \alpha_2 \frac{1 - \tilde{b}_2(\alpha_1)}{\alpha_1} + \right.\\ &+ \left.(\alpha_1(1 - \tilde{b}_1(\alpha_2)) + \alpha_2(1 - \tilde{b}_2(\alpha_1)))b_3\right]\pi_0,\end{aligned}$$

from which the formula (18) follows that proves the theorem.

Remark

For homogeneous system, when $\alpha_i = \alpha, \;\; b_i(x) = b(x) \;\; (i = 1, 2)$, the last expression takes the form

$$\pi_0 = \frac{1}{1 + 2(1 - \tilde{b}(\alpha))(1 + \alpha b_3)},$$

which coincides with previously found.

Quasi-stationary probabilities

Studying the system behavior at its life cycle (during its life time) instead of its stationary probabilities (that all equals to 0 or 1) more interesting are so called **quasi-stationary probability** (q.s.p.) which is defined as limits of conditional probabilities to be in any state given the system is not file yet,

$$\begin{aligned}\hat{\pi}_i &= \lim_{t\to\infty} \mathbf{P}\{J(t)=i|t\le T\} = \\ &= \lim_{t\to\infty} \frac{\mathbf{P}\{J(t)=i,\ t\le T\}}{\mathbf{P}\{t\le T\}} = \lim_{t\to\infty} \frac{\pi_i(t)}{R(t)}.\end{aligned} \tag{19}$$

In order to calculate these limits it is possible to use LT of appropriate functions, which has been given in the section 3 by formulas (3).

$$
\begin{aligned}
\tilde{\pi}_0(s) &= \frac{1}{s+\psi(s)},\\
\tilde{\pi}_1(s) &= \alpha_1 \frac{1-\tilde{b}_1(s+\alpha_2)}{(s+\alpha_2)(s+\psi(s))},\\
\tilde{\pi}_2(s) &= \alpha_2 \frac{1-\tilde{b}_2(s+\alpha_1)}{(s+\alpha_1)(s+\psi(s))},\\
\tilde{\pi}_3(s) &= \frac{\alpha_1\alpha_2(\phi_1(s)+\phi_2(s)}{s(s+\alpha_1)(s+\alpha_2)(s+\psi(s))},\\
\tilde{R}(s) &= \frac{(s+\alpha_1)(s+\alpha_2)+\alpha_1\phi_1(s)+\alpha_2\phi_2(s)}{(s+\alpha_1)(s+\alpha_2)(s+\psi(s))},
\end{aligned}
$$

where the following notations are used

$$
\begin{aligned}
\phi_i(s) &= (s+\alpha_i)(1-\tilde{b}_i(s+\alpha_{i^*})),\ (i=1,2),\\
\psi(s) &= \alpha_1(1-\tilde{b}_1(s+\alpha_2))+\alpha_2(1-\tilde{b}_2(s+\alpha_1)).
\end{aligned}
$$

Theorem

The q.s.p. of the model under consideration have the form

$$\hat{\pi}_i = \lim_{t\to\infty} \frac{\pi_i(t)}{R(t)} = \frac{A_i}{A_R}, \tag{20}$$

where values A_i, A_R are residuals of the functions $\tilde{\pi}_i(s)$ and $\tilde{R}(s)$ in the point $-\gamma$, which is the maximal root of the equation

$$\psi(s) = -s. \tag{21}$$

Proof

For the q.s.p. calculation we use its LT directly.

- Note that the behavior of the functions $\pi_i(t)$ and $R(t)$ for $t \to \infty$ depends on the roots of their LT denominators.
- Note now that the denominators of these functions LT are almost the same, and the behavior of the functions $\pi_i(t)$ and $R(t)$ for $t \to \infty$ depends mostly on the maximal (with minimal absolute value) root.
- The denominator of the function $\tilde{R}(s)$ have only negative roots: $s_1 = -\alpha_1, \; s_2 = -\alpha_2$ and the roots of the equation

$$\psi(s) = -s. \tag{22}$$

For simplicity consider firstly the homogeneous case, when $\alpha_i = \alpha,\ b_i(t) = b(t)$. In this case the function $\psi(x)$ has a form

$$\psi(s) = 2\alpha(1 - \tilde{b}(s + \alpha)).$$

We will consider the solution of this equation for real values of s. For these values the function $\tilde{b}(\cdot)$ is quite monotone one (see [16], vol 2) and therefore is convex, thus the function $1 - \tilde{b}_i(\cdot)$ is concave. It shows that the equation (22) has a unique root, denoted by $-\gamma$ (see fig. 1), which satisfies to the inequality

$$-\alpha < -\gamma < 0.$$

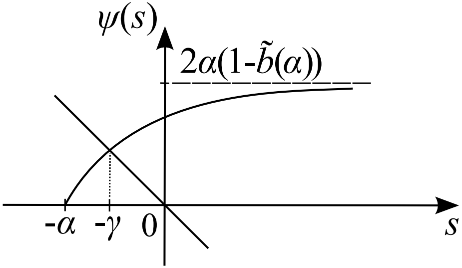

Figure 4: Root of the equation $\psi(s) = 2\alpha(1 - \tilde{b}(s + \alpha))$.

This argumentations show that the functions $\pi_i(t)$ and $R(t)$ have the forms

$$\begin{aligned}
\pi_0(t) &= A_0 e^{-\gamma t}(1+\epsilon_0(t)), \\
\pi_{(1,2)} &\equiv \pi_1(t)+\pi_2(t) = A_{(1,2)} e^{-\gamma t}(1+\epsilon_{(1,2)}(t)), \\
\pi_3(t) &= A_3 e^{-\gamma t}(1+\epsilon_3(t)), \\
R(t) &= A_R e^{-\gamma t}(1+\epsilon_R(t)),
\end{aligned}$$

where functions $\epsilon_i(t)$ $(i = 1, (1,2), 3)$ and $\epsilon_R(t)$ are infinitely small.

These representations allow to calculate quasi-stationary probabilities (19) as follows

$$\hat{\pi}_i = \lim_{t\to\infty} \frac{\pi_i(t)}{R(t)} = \lim_{s\to-\gamma} \frac{\tilde{\pi}_i(s)}{\tilde{R}(s)} = \frac{A_i}{A_R}, \tag{23}$$

where values A_i, A_R are residuals of the functions $\tilde{\pi}_i(s)$ and $\tilde{R}(s)$ in the point $-\gamma$.

An Example

Thus for example for homogeneous case one can get

$$\hat{\pi}_0 = \lim_{s \to -\gamma} \frac{\tilde{\pi}_0(s)}{\tilde{R}(s)} = \frac{\alpha - \gamma}{\alpha - \gamma + 2\alpha(1 - \tilde{b}(\alpha - \gamma))},$$

$$\hat{\pi}_{(1,2)} = \hat{\pi}_1 + \hat{\pi}_2 = \frac{2\alpha(1 - \tilde{b}(\alpha - \gamma))}{\alpha - \gamma + 2\alpha(1 - \tilde{b}(\alpha - \gamma))}.$$

Moreover for the Markov case, when $b(t) = \beta e^{-\beta t}$ these formulas after substitution of appropriate expressions for $\tilde{b}(s + \alpha) = \beta(s + \alpha + \beta)^{-1}$ coincide with appropriate expressions, obtained with the usual Markov approach

$$\hat{\pi}_0 = \frac{\alpha + \beta - \gamma}{3\alpha + \beta - \gamma},$$

$$\hat{\pi}_1 + \hat{\pi}_2 = \frac{2\alpha}{3\alpha + \beta - \gamma}.$$

For investigation of the general heterogeneous case suppose for a certainty that $\alpha_1 < \alpha_2$.

In this case the functions $\tilde{b}_i(s + \alpha_{i^*})$ are also convex, and therefore the functions

$$1 - \tilde{b}_i(s + \alpha_{i^*})$$

as well as their linear combinations are concave and therefore the equation (22) has a unique root, which is also will be denoted as $-\gamma$.

The procedure of the quasi-stationary probabilities is the same as in the above special case.

Sensitivity analysis

All above results show the evident sensitivity of the considered systems characteristics to the shape of their components repair time distribution.

However, under rare failures these sensitivity became negligible for all characteristics of the system under consideration. For reliability function it is shown in another talk at this Conference [15].

Below the asymptotic insensitivity of s.s.p. and q.s.p. of considered systems to the shape of their components repair time distribution under rare failures will be shown.

Partial repair

Remind the s.s.p. of the system under partial failures that is represented by formulas (11).

$$
\begin{aligned}
\pi_1 &= C_1 \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2},\\
\pi_2 &= C_2 \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1},\\
\pi_{(3,1)} &= C_1 b_1 \frac{1-\tilde{b}_1(\alpha_2)}{\alpha_2},\\
\pi_{(3,2)} &= C_2 b_2 \frac{1-\tilde{b}_2(\alpha_1)}{\alpha_1}.
\end{aligned}
\tag{24}
$$

with the values of C_1, C_2, Δ_1, Δ_2, Δ and π_0 given by Corollary 23.

With the help of Tailor expansion for $\max\{\alpha_1, \alpha_2\} \to 0$ the following theorem can be proved, where for the second moment of the repair time the following additional notations are used

$$b_i^{(2)} = \int_0^\infty x^2 b_i(x)\,dx \quad (i = 1, 2)$$

Theorem

Under the rare components' failures, when $\max\{\alpha_1, \alpha_2\} \to 0$ with the notation

$$b_i^{(2)} = \int_0^\infty x^2 b_i(x)dx \ (i = 1, 2).$$

for the second moment of the repair time the following formulas the s.s.p. of the considered system with partial repair take place

$$\begin{aligned}
\pi_0 &\approx \frac{1 - \rho_1\rho_2}{1 + \rho_1 + \rho_2 - 3\rho_1\rho_2}, \\
\pi_i &\approx \frac{\rho_i(1 - \rho_{i^*})}{1 + \rho_1 + \rho_2 - 3\rho_1\rho_2}\left(1 - \frac{b_i^{(2)}\rho_{i^*}}{2b_1 b_2}\right) \quad (i = 1, 2) \\
\pi_{(3,i)} &\approx \frac{\rho_1\rho_2(1 - \rho_{i^*})}{1 + \rho_1 + \rho_2 - 3\rho_1\rho_2}\frac{b_i^{(2)}}{2b_1 b_2}, \quad (i = 1, 2)
\end{aligned} \tag{25}$$

that show their asymptotic insensitivity on the shapes of their components' repair distributions.

Full repair

Analogous to the previous case the s.s.p. of the system under full system repair after its failure represented by formulas (**??**). Tailor expansion of these results up to the second order of the value $\max\{\alpha_1, \alpha_2\} \to 0$ allows to prove following theorem, which show asymptotic insensitivity of the s.s.p. on the shapes of their components' repair distributions, but only on their mean values and the components failure intensities.

Theorem

Under the rare components' failures the s.s.p. of the considered system with full repair take the form

$$
\begin{aligned}
\pi_0 &\approx [1+\rho_1(1+\alpha_2 b_3)+\rho_2(1+\alpha_1 b_3)]^{-1}, \\
\pi_1 &\approx \frac{\rho_1}{1+\rho_1(1+\alpha_2 b_3)+\rho_2(1+\alpha_1 b_3)}, \\
\pi_2 &\approx \frac{\rho_2}{1+\rho_1(1+\alpha_2 b_3)+\rho_2(1+\alpha_1 b_3)}, \\
\pi_3 &\approx \frac{(\rho_1\alpha_2+\rho_2\alpha_1)b_3}{1+\rho_1(1+\alpha_2 b_3)+\rho_2(1+\alpha_1 b_3)}.
\end{aligned}
\tag{26}
$$

Quasi-stationary probabilities

For quasi-stationary probabilities taking into account that $\gamma < \min\{\alpha_1, \alpha_2\}$ with the help of Tailor expansion of the formulas (**??**) in neighbor of the points $\alpha_i - \gamma$ when $\alpha = \max\{\alpha_1, \alpha_2\} \to 0$ one can get the following theorem

Theorem

Under the rare components' failures the q.s.p. of the considered system with full repair take the form

$$\begin{aligned}
\pi_0 &\approx (1+\rho_1+\rho_2)^{-1}, \\
\pi_1 &\approx \frac{\rho_1}{1+\rho_1+\rho_2}, \\
\pi_2 &\approx \frac{\rho_2}{1+\rho_1+\rho_2}.
\end{aligned} \tag{27}$$

Conclusion

- Markovization method is used for heterogeneous double redundant hot standby renewable reliability system analysis.
- The time dependent, stationary and quasi-stationary probability distributions for the system are calculated.
- It was shown that under rare failures the reliability characteristics asymptotically insensitive to the shape of the components repair time distributions up to their two first moments.

Acknowlegements

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