Factorization method in boundary crossing problems for random walks

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Let $X, X_1, X_2, \dots$ be i.i.d. random variables,

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Given a Borel set $B \subset \mathbb{R}$, introduce the first hitting time

$$N = \min\{n \geq 1 :\ \ S_n \in B\}.$$

Put $N = \infty$ if $S_n \in \overline{B} = \mathbb{R} \setminus B$ for all n.

We are interesting in the joint distribution of the pair $(N,\ S_N)$ in the cases: $B = [b, \infty)$, $B = (-\infty, a]$ (one-sided problems, $a < 0,\ b > 0$), $B = (-\infty, a] \cup [b, \infty)$ (two-sided problem).

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We will demonstrate an analytical method to study these distributions.

Introduce the double Laplace–Stieltjes transform (LST)

$$Q(z,\lambda) = \mathbf{E}\big(z^N e^{\lambda S_N};\ N < \infty\big)$$

$$= \sum_{n=1}^{\infty} z^n \int_B e^{\lambda y} \mathbf{P}(N = n,\, S_N \in dy),$$

and, in addition, the functions $\varphi(\lambda) = \mathbf{E}e^{\lambda X}$ and

$$Q_0(z,\lambda) = \sum_{n=1}^{\infty} z^n \mathbf{E}\big(e^{\lambda S_n};\ N > n\big).$$

At the first step, our goal is to find $Q(z,\lambda)$.

The following assertion (the main identity) is known (W.Feller, Vol.2, Ch.18).

Theorem

For $|z| < 1$ *and* $\operatorname{Re}\lambda = 0$ *the following identity holds:*

$$\big(1 - z\varphi(\lambda)\big)\big(1 + Q_0(z,\lambda)\big) = 1 - Q(z,\lambda).$$

So we have one equation containing two unknown functions. Nevertheless, we can solve it and find the functions $Q(z\lambda)$ and $Q_0(z\lambda)$ in one-sided and two-sided problems, but, to this end, we need factorization of the function $1 - z\varphi(\lambda)$.

It can be easily shown that, for $|z| < 1$ and $\mathrm{Re}\,\lambda = 0$

$$1 - z\varphi(\lambda) = \exp\{\log(1 - z\varphi(\lambda))\} = R_-(z,\lambda)R_+(z,\lambda)$$

(Wiener–Hopf factorization), where

$$R_-(z,\lambda) = \exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{E}\left(\exp\{\lambda S_n\}; S_n \le 0\right)\right\},$$

$$R_+(z,\lambda) = \exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{E}\left(\exp\{\lambda S_n\}; S_n > 0\right)\right\}.$$

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The function $R_+(z,\lambda)$ is analytic with respect to λ in the left half-plane $\operatorname{Re}\lambda < 0$, continuous at the border, and it is bounded and does not equal zero when $\operatorname{Re}\lambda \le 0$. The function $R_-(z,\lambda)$ has similar properties in the right half-plane. The components of a factorization with the above properties are defined uniquely up to a constant factor.

There exist some other expressions for $R_\pm(z,\lambda)$.

Denote $S(A)$ the set of functions g taking the form

$$g(\lambda) = \int_A e^{\lambda y} dG(y), \quad \text{where} \quad \int_A |dG(y)| < \infty, \quad \operatorname{Re}\lambda = 0.$$

We notice, in addition, that the functions $R_+(z,\lambda)$, $R_+^{-1}(z,\lambda)$ belong to $S\big([0,\infty)\big)$, and the functions $R_-(z,\lambda)$, $R_-^{-1}(z,\lambda)$ belong to $S\big((-\infty,0]\big)$.

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Given a function $g \in S(\mathbb{R})$, we define

$$[g(\lambda)]^A = \int_A e^{\lambda y} dG(y)$$

for each Borel set A.

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$$Q(z,\lambda) = R_+(z,\lambda)\left[R_+^{-1}(z,\lambda)\right]^{[b,\infty)}, \qquad \text{if } B = [b,\infty), \quad b > 0,$$

$$Q(z,\lambda) = R_-(z,\lambda)\left[R_-^{-1}(z,\lambda)\right]^{(-\infty,a]} \quad \text{if } B = (-\infty,a], \quad a < 0.$$

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Proof. Let $B = [b,\infty)$, $b > 0$. Using factorization, we rewrite the main identity:

$$R_-(z,\lambda)R_+(z,\lambda)\big(1 + Q_0(z,\lambda)\big) = 1 - Q(z,\lambda),$$

and then redistribute summands between left-hand and right-hand sides:

$$R_-(z,\lambda)Q_0(z,\lambda) = -R_-(z,\lambda) + R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big).$$

The left-hand side of this relation belongs to $S\big((-\infty,b)\big)$, so the same is true for the right-hand side, i.e.

$$\Big[-R_-(z,\lambda) + R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big)\Big]^{[b,\infty)} \equiv 0.$$

Clearly, $\big[R_-(z,\lambda)\big]^{[b,\infty)} \equiv 0.$ Further, under our condition, $Q(z,\lambda) \in S([b,\infty))$, so $R_+^{-1}(z,\lambda)Q(z,\lambda) \in S\big([b,\infty)\big)$.
Hence,

$$\Big[R_+^{-1}(z,\lambda)\big(1 - Q(z,\lambda)\big)\Big]^{[b,\infty)}$$

$$= \Big[R_+^{-1}(z,\lambda)\Big]^{[b,\infty)} - R_+^{-1}(z,\lambda)Q(z,\lambda) = 0.$$

A symmetric reasoning establishes the result for $B = (-\infty, a], \quad a < 0.$ The theorem is proved.

Next we introduce operators $\mathcal{L}_{\pm}$. Given a function $g \in S(\mathbb{R})$, we put

$$(\mathcal{L}_{-}g)(z,\lambda) = R_{-}(z,\lambda)\left[R_{-}^{-1}(z,\lambda)g(\lambda)\right]^{(-\infty,a]},$$

$$(\mathcal{L}_{+}g)(z,\lambda) = R_{+}(z,\lambda)\left[R_{+}^{-1}(z,\lambda)g(\lambda)\right]^{[b,\infty)}.$$

Here $|z| < 1$, $\operatorname{Re}\lambda = 0$, the function g may also depend on z.

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$$(\mathcal{L}_- g)(z,\lambda) = R_-(z,\lambda)\big[R_-^{-1}(z,\lambda)g(\lambda)\big]^{(-\infty,a]},$$

$$(\mathcal{L}_+ g)(z,\lambda) = R_+(z,\lambda)\big[R_+^{-1}(z,\lambda)g(\lambda)\big]^{[b,\infty)}.$$

Here $|z|<1$, $\operatorname{Re}\lambda = 0$, the function g may also depend on z.

Put $e(\lambda) = e(z,\lambda) \equiv 1$. In the new notations, the formulas obtained above can be rewritten in the following way:

$$Q(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) \ \text{ if } \ B = [b,\infty),$$

$$Q(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) \ \text{ if } \ B = (-\infty,a].$$

It turns out that the double LST in the two-sided problem can be also expressed via operators $\mathcal{L}_\pm$.
Really, put $B = (-\infty, a] \cup [b, \infty)$ then

$$N = \min\big\{n \geq 1 : \ S_n \notin (a,\, b)\big\}, \quad a < 0, \ b > 0.$$

Let

$$Q_1(z,\lambda) = \mathbf{E}\big(z^N e^{\lambda S_N};\ S_N \leq a\big),\ \ Q_2(z,\lambda) = \mathbf{E}\big(z^N e^{\lambda S_N};\ S_N \geq b\big).$$

Then $Q(z,\lambda = Q_1(z,\lambda) + Q_2(z,\lambda)$.

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Then $Q(z, \lambda = Q_1(z,\lambda) + Q_2(z,\lambda)$.
In the same way as in Theorem 2, from the main identity we obtain

$$Q_2(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) - (\mathcal{L}_+ Q_1)(z,\lambda), \qquad (1)$$
$$Q_1(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) - (\mathcal{L}_- Q_2)(z,\lambda).$$

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Substituting the expression for $Q_1(z,\lambda)$ into (1) leads to the identity

$$Q_2(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) - (\mathcal{L}_+\mathcal{L}_- e)(z,\lambda) + (\mathcal{L}_+\mathcal{L}_- Q_2)(z,\lambda),$$

and, in a similar way, we arrive at the identity for Q_1.

Further, for a random walk with nonzero drift, consider the random variable η equal to the number of upcrossings of the strip with boundaries at the levels $a < 0$ and $b > 0$. It turns out that, in this case,

$$\mathbf{P}(\eta \geq k) = \lim_{z \to 1} \Big((\mathcal{L}_+ \mathcal{L}_-)^k e \Big)(z, 0), \quad k \geq 1.$$

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Thus, we see that, in many boundary crossing problems connected with the achievement of a set with linear boundaries, LST of the distributions under study are expressed in terms of the operators $\mathcal{L}_\pm$. So, we need to clarify the probabilistic meaning of these operators, as well as the possibility of finding explicit expressions for them and asymptotic representations.

Discuss a probabilistic meaning.
First, it is not difficult to deduce from the main identity that

$$Q(z,\lambda) = 1 - R_+(z,\lambda) \text{ if } B = (0,\infty),$$
$$Q(z,\lambda) = 1 - R_-(z,\lambda) \text{ if } B = (-\infty,0].$$

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In both of these cases the function $Q(z,\lambda)$ is a joint distribution of the corresponding ladder epoch and ladder height of the random walk. Thus, using factorization components for finding the LST of distributions of boundary functionals is not some technical trick. It means that the desired distributions are expressed in terms of the distributions of ladder epoch and ladder height, which is quite natural.

Further, let $\tau \geq 0$ be an arbitrary stopping time, possibly improper. At the event $\{\tau < \infty\}$, we define the random variables

$$\tau_+(b) = \inf\{n \geq \tau : \ S_n \geq b\}, \quad \tau_-(a) = \inf\{n \geq \tau : \ S_n \leq a\}.$$

Suppose that the double transform $f(z,\lambda) = \mathbf{E}\big(z^\tau \exp\{\lambda S_\tau\}; \ \tau < \infty\big)$ is known. The problem is to find the functions

$$f_+(z,\lambda) = \mathbf{E}\big(z^{\tau_+(b)} \exp\{\lambda S_{\tau_+(b)}\}; \ \tau_+(b) < \infty\big),$$

$$f_-(z,\lambda) = \mathbf{E}\big(z^{\tau_-(a)} \exp\{\lambda S_{\tau_-(a)}\}; \ \tau_-(a) < \infty\big).$$

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The following assertion was obtained in Lotov'89.

Theorem

For $|z| < 1$ and $\mathrm{Re}\,\lambda = 0$, the following relations hold:

$$f_\pm(z, \lambda) = (\mathcal{L}_\pm f)(z, \lambda).$$

The assertion of this theorem makes clear the probabilistic meaning of all summands in the relations

$$Q_2(z,\lambda) = (\mathcal{L}_+ e)(z,\lambda) - (\mathcal{L}_+ Q_1)(z,\lambda),$$

$$Q_1(z,\lambda) = (\mathcal{L}_- e)(z,\lambda) - (\mathcal{L}_- Q_2)(z,\lambda).$$

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We note in passing that, under the conditions of the theorem, the distributions of jumps of a walk to the time τ and after it may not coincide. This makes it possible to consider random walks in which the distribution of jumps is changing at the moment of passing certain boundaries.

Next, we discuss the possibilities of calculating the factorization components and operators $\mathcal{L}_\pm$ in an explicit form. The explicit form of the factorization components is known for Gaussian random walks (Lotov'96) and for walks for which the function $\mathbf{E}\big(\exp\{\lambda X\};\, X<0\big)$ or $\mathbf{E}\big(\exp\{\lambda X\};\, X>0\big)$ is rational (Borovkov'72). For example, if the function

$$\mathbf{E}\big(\exp\{\lambda X\};\, X>0\big) = \frac{R(\lambda)}{P(\lambda)}, \quad \text{where} \quad P(\lambda) = \prod_{i=1}^{k}(\lambda - p_i),$$

is rational then

$$R_+(z,\lambda) = \frac{\Lambda(z,\lambda)}{P(\lambda)}, \qquad R_-(z,\lambda) = \frac{(1-z\varphi(\lambda))P(\lambda)}{\Lambda(z,\lambda)},$$

where $\Lambda(z,\lambda) = \prod_{j=1}^{k}(\lambda - \lambda_j(z))$, and $\lambda_1(z), \ \ldots, \ \lambda_k(z)$ are zeros of the function $1 - z\varphi(\lambda)$ in the right half-plane (with considering their multiplicities). In this case the calculation of $(\mathcal{L}_+ g)(z,\lambda)$ becomes a simple exercise if the function $R_+^{-1}(z,\lambda)$ is first decomposed on simple fractions.

Let, in particular,

$$\mathbf{P}(X \geq t) = q \exp\{-\alpha t\}, \quad t \geq 0.$$

Denote $\lambda(z)$ the only positive solution of the equation $1 - z\varphi(\lambda) = 0$, then

$$R_+(z, \lambda) = \frac{\lambda - \lambda(z)}{\lambda - \alpha},$$

and for each function $g \in S((-\infty, 0])$ we have

$$(\mathcal{L}_+ g)(z, \lambda) = \frac{\lambda(z) - \alpha}{\lambda - \alpha} g(\lambda(z)) e^{(\lambda - \lambda(z))b}.$$

Let us now investigate the asymptotic behavior of the operators $\mathcal{L}_{\pm}$ as $a \to -\infty, \;\; b \to \infty$.
We assume here:
(A) the distribution of X contains an absolutely continuous component;
(C) the Cramér condition:

$$\varphi(\lambda) < \infty \;\; \text{for} \; -\gamma \le \lambda \le \beta, \; \gamma > 0, \; \beta > 0.$$

In addition, we assume that $\mathbf{E}e^{\beta X} > 1$ if $\mathbf{E}X < 0$ and $\mathbf{E}e^{-\gamma X} > 1$ if $\mathbf{E}X > 0$.

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Under these conditions, one can distinguish the principal terms of the asymptotics for $(\mathcal{L}_\pm g)(z,\lambda)$ as $a \to -\infty$, $b \to \infty$ and estimate the remainders that turn out to be exponentially small in comparison with the principal terms (Lotov'99).

The following representations hold uniformly in $z \in (1-\delta, 1)$ for some $\delta > 0$ and $\varepsilon > 0$:

$$(\mathcal{L}_+ g)(z,\lambda) = v_z(\lambda) e^{(\lambda - \lambda_+(z))b} g(\lambda_+(z)) \big(1 + O\big(e^{-\varepsilon b}\big)\big)$$

for each $g \in S((-\infty, 0])$, and

$$(\mathcal{L}_- g)(z,\lambda) = u_z(\lambda) e^{(\lambda_-(z) - \lambda)a} g(\lambda_-(z)) \big(1 + O\big(e^{\varepsilon a}\big)\big)$$

for each $g \in S([0, \infty))$, where $\lambda_-(z) < 0 < \lambda_+(z)$ are zeros of the function $1 - z\varphi(\lambda)$ and

$$v_z(\lambda) = \frac{R_+(z,\lambda)}{(\lambda - \lambda_+(z)) R'_+(z, \lambda_+(z))},$$

$$u_z(\lambda) = \frac{R_-(z,\lambda)}{(\lambda - \lambda_-(z)) R'_-(z, \lambda_-(z))}.$$

As a result, for $B = [b, \infty)$, $b \to \infty$, we find

$$\mathbf{E}\big(z^N e^{\lambda S_N}; S_N \geq b\big) = v_z(\lambda) e^{(\lambda - \lambda_+(z))b}\big(1 + O\big(e^{-\varepsilon b)}\big)\big).$$

The remainder $O\big(e^{-\varepsilon b)}\big)$ vanishes if

$$\mathbf{P}(X \geq t) = q \exp\{-\alpha t\}, \quad t \geq 0.$$

For the two-sided boundary crossing problem when $B = (-\infty, a] \cup [b, \infty)$, we obtain, as $a \to -\infty$, $b \to \infty$,

$$\mathbf{E}\big(z^N e^{\lambda S_N}; S_N \geq b\big)$$

$$= v_z(\lambda) e^{(\lambda - \lambda_+(z))b} \frac{1 - v_2(z)\mu^a(z)}{1 - v_1(z) v_2(z) \mu^{a+b}(z)} \big(1 + O\big(e^{-\varepsilon b)}\big)\big)$$

uniformly in $z \in (1 - \delta, 1)$ for some $\delta > 0$, where

$$v_1(z) = v_z(\lambda_-(z)),\ v_2(z) = u_z(\lambda_+(z)),\ \mu(z) = e^{\lambda_-(z) - \lambda_+(z)}.$$

The principle terms of these asymptotic representations can be easily inverted with respect to spatial variable λ in both one-sided and two-sided problems. Putting $z = 1$, we come to the asymptotic representations for $\mathbf{E}\big(e^{\lambda S_N}; S_N \geq b\big)$ and then we come to the exact and asymptotic formulas for the distribution of overshoot in the one-sided boundary problem, for $\mathbf{E}N$ and for the ruin probability $\mathbf{P}(S_N \geq b)$ in the two-sided problem, for the distribution of the number of crossings of the strip, etc.

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We present here some results.

Theorem

Suppose that conditions (A) and (C) hold, $\mathbf{E}\,X = 0$ *and* $B = (-\infty, a] \cup [b, \infty)$. *Then, as* $b - a \to \infty$, *for each* $y \geq 0$,

$$\mathbf{P}(S_N \geq b + y) = A(y)\frac{c_1 - a}{c_2 + b - a} + O\left(e^{-\varepsilon(b+y)}\right),$$

where $\varepsilon > 0$, c_1 *and* c_2 *are the constants expressed via first two moments of ladder heights and* $\mathbf{E}X^2$,

$$A(y) = \mathbf{E}\,(\chi - y;\ \chi \geq y)(\mathbf{E}\,\chi)^{-1},$$

$$\chi = S_\nu, \quad \nu = \min\{n \geq 1 :\ S_n > 0\}.$$

Similar relation holds for $\mathbf{P}(S_N \leq -a - y)$. Further applying Wald's identity $\mathbf{E}\,S_N^2 = \mathbf{E}\,N\,\mathbf{E}\,X^2$ leads to the asymptotic expansion for $\mathbf{E}N$ up to $O\left(e^{\varepsilon a}\right) + O\left(e^{-\varepsilon b}\right)$.

Theorem

Suppose that conditions (A) and (C) hold, $\mu_1 = \mathbf{E}\, X_1 < 0$, *and* $B = (-\infty, a] \cup [b, \infty)$. *Then, as* $a \to -\infty$, $b \to \infty$,

$$\mu_1 \mathbf{E}\, N = a - K_1 + ((b - a + K_1)K_2 + K_3)e^{-qb}$$

$$+ O\big(e^{-(q+\varepsilon)b} + e^{(q+\varepsilon)a}\big)\big)$$

where K_i *are known constants,* $\varphi(q) = 1$, $q > 0$, *and* $\varepsilon > 0$.

The inversion of the principle terms in the time variable z is more complicated. Nevertheless, the complete asymptotic expansions for the probabilities $\mathbf{P}(N = n, S_N \geq x)$ were obtained in one-sided boundary problem (A.A. Borovkov) and in two-sided boundary problem (V.I. Lotov). In both cases special modification of the saddle point method was applied under condition that $n \to \infty$, $b = b(n) \to \infty$, $a = a(n) \to -\infty$, and $b - a = o(n)$.

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Factorization method for boundary crossing problems was first derived for random walks (A.A. Borovkov, B.A. Rogozin, V.I. Lotov, D.K. Kim), then for stochastic processes with independent increments (B.A. Rogozin, V.R. Khodjibayev), for Markov modulated random walks (E.L. Presman, V.I. Lotov).

Thank you for attention!