INTEGRO-LOCAL LIMIT THEOREMS FOR MULTIDIMENSIONAL COMPOUND RENEWAL PROCESSES

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Formulation of the problem

Let $(\tau, \zeta), (\tau_1, \zeta_1), (\tau_2, \zeta_2), \cdots$ — be a sequence i.i.d. random vectors,

$$\tau > 0, \quad \zeta := (\zeta_{(1)}, \cdots, \zeta_{(d)}) \in \mathbb{R}^d, \quad d \geq 1.$$

Let

$$T_0 := 0, \quad T_{n+1} := T_n + \tau_{n+1}, \quad \mathbf{Z_0} := \mathbf{0}, \quad \mathbf{Z}_{n+1} := \mathbf{Z}_n + \zeta_{n+1};$$

$$\nu(t) := \max\{k \geq 0 : \; T_k < t\},$$
$$\eta(t) := \min\{k \geq 0 : \; T_k \geq t\}.$$

The compound renewal processes $\mathbf{Z}(t)$, $\mathbf{Y}(t)$ for the sequence (τ_j, ζ_j), $j \geq 1$, are defined as (см. [1],[2]):

$$\mathbf{Z}(t) := \mathbf{Z}_{\nu(t)}, \quad \mathbf{Y}(t) := \mathbf{Z}_{\eta(t)}, \quad t \geq 0.$$

Formulation of the problem

Let the Crame'r moment condition hold:
$[\mathbf{C}_0]$. *For some* $\delta > 0$

$$\mathbf{E}e^{\delta\tau+\delta|\zeta|} < \infty.$$

For a vector $\mathbf{x} = (x_{(1)}, \cdots, x_{(d)}) \in \mathbb{R}^d$ let

$$\Delta[\mathbf{x}) := \prod_{j=1}^{d} [x_{(j)}, x_{(j)} + \Delta), \quad \Delta > 0.$$

We study integro-local limit theorems for $\mathbf{Z}(T)$, $\mathbf{Y}(T)$, i.e. the exact asymptotics for the probabilities

$$\mathbf{P}\big(\mathbf{Z}(T) \in \Delta[\mathbf{x})\big) =?, \quad \mathbf{P}\big(\mathbf{Y}(T) \in \Delta[\mathbf{x})\big) =?$$

in the range of normal and large deviations.
This is joint work with E.I.Prokopenko.

Deviation Function

For $(\lambda, \boldsymbol{\mu}) = (\lambda, \mu_{(1)}, \cdots, \mu_{(d)}) \in \mathbb{R}^{d+1}$ let us define

$$\mathbf{A}(\lambda, \boldsymbol{\mu}) := \ln \mathbf{E} e^{\lambda\tau + \boldsymbol{\mu\zeta}},$$

where $\boldsymbol{\mu\zeta} := \mu_{(1)}\zeta_{(1)} + \cdots + \mu_{(d)}\zeta_{(d)}$;

$$\mathcal{A}^{\leq 0} := \{(\lambda, \boldsymbol{\mu}) : \ \mathbf{A}(\lambda, \boldsymbol{\mu}) \leq 0\}.$$

Let

$$\lambda_+ := \sup\{\lambda : \quad \mathbf{E} e^{\lambda\tau} < \infty\}.$$

Then for $\boldsymbol{\mu} \in \mathbb{R}^d$ put

$$\mathbf{A}(\boldsymbol{\mu}) := -\sup\{\lambda : \ (\lambda, \boldsymbol{\mu}) \in \mathcal{A}^{\leq 0}\},$$

where $\sup\{\lambda : \quad \lambda \in \emptyset\} = -\infty$;

$$\widehat{\mathbf{A}}(\boldsymbol{\mu}) := \max\{\mathbf{A}(\boldsymbol{\mu}), \ -\lambda_+\},$$

Deviation function

Finally, for $\boldsymbol{\alpha} \in \mathbb{R}^d$ define

$$\mathbf{D}(\boldsymbol{\alpha}) := \sup_{\boldsymbol{\mu}} \{\boldsymbol{\mu}\boldsymbol{\alpha} - \mathbf{A}(\boldsymbol{\mu})\},$$

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Definition 1 (LDP)

Processes $\left\{\frac{\mathbf{Z}(T)}{T}\right\}$ *satisfies the large deviation principle (LDP) in* $\mathbb{R}^d$ *with good rate function* $\widehat{\mathbf{D}}(\boldsymbol{\alpha})$, *if for all* $B \subset \mathbb{R}^d$

$$\limsup_{T\to\infty} \frac{1}{T} \ln \mathbf{P}\left(\frac{\mathbf{Z}(T)}{T} \in B\right) \leqslant -\inf_{\alpha\in[B]} \widehat{D}(\alpha),$$

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}\left(\frac{\mathbf{Z}(T)}{T} \in B\right) \geqslant -\inf_{\alpha\in(B)} \widehat{D}(\alpha),$$

the set $\left\{\boldsymbol{\alpha} \in \mathbb{R}^d \ : \ \widehat{D}(\alpha) \leqslant c\right\}$ $-$ *is compact for all* $c \in \mathbb{R}$.

The Large Deviation Principle

Theorem 1 ([2])

Assume

$$\varliminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}(\tau \geq T) \geq -\lambda_+. \qquad (1)$$

Then processes $\left\{\frac{Z(T)}{T}\right\}$ *satisfies LDP in* $\mathbb{R}^d$ *with good rate function* $\widehat{\mathbf{D}}(\boldsymbol{\alpha})$, *also*

$$\lim_{T\to\infty} \frac{1}{T} \ln \mathbf{E} e^{\boldsymbol{\mu}\mathbf{Z}(T)} = \widehat{\mathbf{A}}(\boldsymbol{\mu}), \quad \mu \in \mathbb{R}^d.$$

If, additionally,

$$\lambda_+ \geq \mathbf{D}(\mathbf{0})$$

hold, then

$$\widehat{\mathbf{A}}(\boldsymbol{\mu}) = \mathbf{A}(\boldsymbol{\mu}), \quad \widehat{\mathbf{D}}(\boldsymbol{\alpha}) = \mathbf{D}(\boldsymbol{\alpha}),$$

and condition (1) can be omitted.

In case $d = 1$, $\lambda_+ \geq \mathbf{D}(\mathbf{0})$, Theorem 1 was established in [1].

Let us define $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\alpha}): \ \mathbb{R}^d \to \mathbb{R}^d$ so that

$$\mathbf{D}(\boldsymbol{\alpha}) = \boldsymbol{\mu}(\boldsymbol{\alpha})\boldsymbol{\alpha} - \mathbf{A}(\boldsymbol{\mu}(\boldsymbol{\alpha})).$$

Then $(\lambda(\boldsymbol{\alpha}), \mu(\boldsymbol{\alpha})) := (-\mathbf{A}(\mu(\boldsymbol{\alpha})), \ \mu(\boldsymbol{\alpha}))$ belongs to the boundary $\partial\mathcal{A}^{\leq 0}$ of the set $\mathcal{A}^{\leq 0}$. Let

$$\mathfrak{A} := \left\{\boldsymbol{\alpha} \in \mathbb{R}^d : \ \ (\lambda(\boldsymbol{\alpha}), \boldsymbol{\mu}(\boldsymbol{\alpha})) \in (\mathcal{A})\right\},$$

where

$$\mathcal{A} := \left\{(\lambda, \boldsymbol{\mu}) : \ \ \mathbf{A}(\lambda, \boldsymbol{\mu}) < \infty\right\};$$

$$\mathfrak{T} := \left\{\boldsymbol{\alpha} \in \mathbb{R}^d : \ \ \lambda(\boldsymbol{\alpha}) \geq \lambda_+\right\}.$$

Theorem 2 ([3])

Let $\boldsymbol{\alpha}^0 \in \mathfrak{A} \setminus \mathfrak{T}$, $\boldsymbol{\alpha} := \frac{\mathbf{x}}{T} \to \boldsymbol{\alpha}^0$ *при* $T \to \infty$. *Then*

$$\mathbf{P}\big(\mathbf{Z}(T) \in \Delta[\mathbf{x})\big) = \frac{\Delta^d}{T^{d/2}} I_Z(\boldsymbol{\alpha}) e^{-T\mathbf{D}(\boldsymbol{\alpha})}(1 + o(1)).$$

Theorem 2 in case $d = 1$ was established in [1].

The exact asymptotic in non-regular case

$[\mathbf{F}_\tau]$. *For all* $t > 0$

$$\mathbf{P}(\tau \geq t) = L(t)e^{-\lambda_+ t + ct^\gamma},$$

where $\gamma \in [0, 1)$, $c \in \mathbb{R}$, $L(t)$ — *regularly varying as* $t \to \infty$ *function.*

Theorem 3 ([3])

Assume $[\mathbf{F}_\tau]$ *hold. Let* $\kappa := \left[\frac{1}{1-\gamma}\right]$; $\boldsymbol{\alpha}^0 \in \mathfrak{A} \cap (\mathfrak{T})$, $\boldsymbol{\alpha}^0 \neq \mathbf{0}$, $\boldsymbol{\alpha} := \frac{\mathbf{x}}{T} \to \boldsymbol{\alpha}^0$ *as* $T \to \infty$. *Then*

$$\mathbf{P}\big(\mathbf{Z}(T) \in \Delta[\mathbf{x})\big) =$$

$$\frac{\Delta^d L(T)}{T^{(d-1)/2}} C(\boldsymbol{\alpha}) e^{-T\widehat{D}(\boldsymbol{\alpha}) - \sum_{k=1}^{\kappa} T^{k\gamma-(k-1)} g_k(\boldsymbol{\alpha})} (1 + o(1)).$$

Let

$$\mathfrak{C} := \{\boldsymbol{\alpha} \in \mathbb{R}^d : \ \ \mu(\boldsymbol{\alpha}) \in (\mathcal{M})\},$$

where

$$\mathcal{M} := \{\boldsymbol{\mu} \in \mathbb{R}^d : \ \mathbf{E}e^{\boldsymbol{\mu}\boldsymbol{\zeta}} < \infty\}.$$

Theorem 4 ([3])

Let $\boldsymbol{\alpha}^0 \in \mathfrak{A} \cap (\mathfrak{C})$, $\boldsymbol{\alpha} := \frac{\mathbf{x}}{T} \to \boldsymbol{\alpha}^0$ *as* $T \to \infty$*. Then*

$$\mathbf{P}\big(\mathbf{Y}(T) \in \Delta[\mathbf{x})\big) = \frac{\Delta^d}{T^{d/2}} I_Y(\alpha) e^{-T\mathbf{D}(\boldsymbol{\alpha})}(1 + o(1)).$$

Theorem 4 in case $d = 1$,

$$\boldsymbol{\alpha}^0 \in (\mathfrak{A} \setminus \mathfrak{T}) \cap (\mathfrak{C})$$

was established in [1].

Finite-dimensional distributions

Let

$$U_0 := 0, \quad U_1 \to \infty, \cdots, U_m \to \infty,$$
$$W_j := U_0 + \cdots + U_j,$$

In [3] were found fairly extensive conditions, under which for

$$\boldsymbol{\alpha}_0 := \mathbf{0}, \quad \boldsymbol{\alpha}_j := \frac{\mathbf{x}_j}{U_j} \to \boldsymbol{\alpha}_j^0, \text{ as } U_j \to \infty, \quad j = 1, \cdots, m,$$

holds

$$\mathbf{P}\left(\cap_{j=1}^m \{\mathbf{Z}(U_j + W_{j-1}) - \mathbf{Z}(W_{j-1}) \in \Delta_j[\mathbf{x}_j)\}\right) =$$
$$\prod_{j=1}^m \frac{\Delta_j^d}{U_j^{d/2}} I_Z(\boldsymbol{\alpha}_{j-1}, \boldsymbol{\alpha}_j) e^{-U_j \mathbf{D}(\boldsymbol{\alpha}_j)}(1 + o(1)). \qquad (2)$$

In article [1], in case $d = 1$ proposed conditions , under which (2) hold.

[1] *Borovkov A. A., Mogulskii A. A.* Integro-local limit theorems for compound renewal processes under Cramér condition.I,II. (to appear)

[2] *Mogulskii A. A., Prokopenko E. I.* Large deviations principle in phase space of the multidimensional compound renewal processes under Cramér condition (to appear)

[3] *Mogulskii A. A., Prokopenko E. I.* Integro-local limit theorems for multidimensional compound renewal processes under Cramér condition.I,II,III. (to appear)

THANK YOU FOR ATTENTION!