Mathematical Methods in Practical Mechanics: From Heron of Alexandria to Galileo

Evgeny Zaytsev

Two provisions were laid at the basis of the new science of the 16th century. According to the first, the world is a kind of machine, set in motion by the “Divine Clockmaker”; according to the second, the movement of this “world machine” (machina mundi) is subject to mathematical laws. Although elements of such representations can also be found at an earlier period, only in the Modern Times they have become normative for defining the methodology of scientific knowledge.

If the ideas of mechanism, which played such an important role in the formation of classical natural science of the 17th century, have lost nowadays their general validity, the idea of mathematical description of the movements of machines and mechanisms still retains its significance. In particular, there is no doubt that the movement of so called “simple machines” (lever, wheel and axle, compound pulley, inclined plane, wedge and screw) obeys the mathematical laws derived from the law of lever. However, until the 16th century, the idea of mathematizing the motion of simple machines was by no means obvious. The history of the formation of this idea will be considered in this article.

Any development has its source in a contradiction, which is removed by this development. The basic contradiction in mechanics of pre-classical period was the fact that the basis of the dynamics of motion of “simple machines” was constituted by statics, viz. by equilibrium of the balance. In classical mechanics, the opposition between dynamics and statics is removed due to common laws describing both the real motion in a dynamical situation and the virtual motion in a static one. In pre-classical mechanics, the existence of such laws was problematic due to the idea of the absolute opposition of motion and rest. This metaphysical postulate excluded the possibility of finding an intermediary link that would have properties of both opposing terms – movement and rest. The absence of an “intermediary” between movement and rest led to the fact that any attempt to transfer the methods of statics to the domain of dynamics should have been a priori qualified as unscientific.

In contrast to the natural world or, more accurately, to the conceptual universe of natural philosophy, in which there was no place for a mediating link between movement and rest, in the field of practical crafts (Greek “techne,” Latin “ars”), such a link has long existed. That was a wheel, in the motion of which both dynamic and static moments were realized. This fact was, of course, known to ancient authors. It was pointed out, in particular, by the author of “Mechanical Problems”: “it is a very great marvel that contraries should be present together, and the circle is made up of contraries. For to begin with, it is formed by motion and rest, things which are by nature opposed to one another.” [Aristotle, 1913, 847b 18-21].

“Mechanical problems” by pseudo-Aristotle (3rd c. BC) is the earliest surviving treatise on practical mechanics. This short book is at the origin of a series of ancient treatises on the technical (mechanical) movement which included: the 10th Book of “On architecture” by Vitruvius (1st c. BC), works on mechanics by Heron of Alexandria (1st c. AD) and the 8th Book of “Mathematical Collection” by Pappus of Alexandria (3rd c. AD).

In “Mechanical Problems,” all technical motions are reduced to the movement of the lever, which in turn is reduced to the balance, and the balance to a circular motion. Thus, as a basic model of technical movement, the movement of the wheel is tacitly used. The wheel does not belong, however, to natural things, that is, it does not contain the “principles” (viz., causes or purposes) of its movement and rest. Since the movements of the wheel are determined by the external aims, set down by the person who created it, the knowledge of the wheel, as well as other artifacts, was only of a tentative character, according to the Aristotelian school. To hint at it, the author of “Mechanical Problems” anticipated his answers to questions concerning the properties of technical motion, with
modal statements “is it because ....” or “does the reason lie in ...

“Mechanical problems” are characterized by the lack of attempts of quantitatively estimate the technical motion. The only exception is question 3, the answer to which contains a brief comment that can be interpreted as an indication of the description of the movement of the lever in terms of proportion. The question is put as follows: “Why is it that ... the exercise of little force raises great weights with the help of a lever...?” The answer to it, anticipated by the usual modal turnover, constitutes, as it were, an attempt to explain the dynamics of motion out of the statics of the weights in balance. “Does the reason lie in the fact that .... the lever acts like the beam of a balance with the cord attached below and divided into two unequal parts?” The meaning of the subsequent reasoning consists in reducing the movement of the lever to the law of equilibrium of weights, expressed in the form of an inversely proportional relationship between the weights on the scales and their distance from the fulcrum or suspension point. “As the weight moved is to the weight moving it, so, inversely, is the length of the arm bearing the weight to the length of the arm nearer to the power. The further one is from the fulcrum, the more easily will one raise the weight...” [Aristotle, 1913, 850b1-3].

The absence of any detailed discussion of the mathematical content of the problem, as well as of specific numerical examples that could illustrate the general principle, indicate that the description of the lever (that is, the most important of the “simple machines”) in terms of numerical quantity was not the task of the author. As mentioned above, when discussing other types of technical movement, he totally avoided accurate quantitative estimates. The same tactics were inherent in other ancient authors of mechanics. Describing the movements of “simple machines,” they usually limited themselves only to a qualitative comparison of the efforts applied to the machine, and their results, without specifying the quantitative parameters of the motion (the relationships between the values of forces, distances, travel time, etc.). So, about the lever, it was usually said that the larger the leverage of the driving force application, the greater will be the gain in force. Similarly, for the wheel and axle it was claimed that the gain in lift will increase following the augmentation of the ratio of the diameter of the wheel to which the force is applied, to the diameter of the axle on which the rope lifting the load is wound. Concerning the compound pulley (polyspast), it was simply noted that the increase in lift is due to the increase in the number of blocks. In the Western European medieval treatises on mechanics, the question of the operation by means of “simple machines” was not discussed at all (except attempts to describe the equilibrium of weights on inclined planes in the school of Jordanus Nemorarius [Zaytsev, 2016]).

Apart from the general tradition, treating the motion of “simple machines” in rough qualitative terms, stands a treatise “Mechanics” by Heron of Alexandria, more precisely, its second, theoretical part [Héron d'Alexandrie, 1894; Schiefsky, 2008]. As for its first, practical part, it does not essentially differ from other ancient treatises on mechanics. Thus, considering the lever, Heron only points out that “the nearer the fulcrum is to the load, the more easily the weight is moved” (2.2.). Similarly, about the wedge, “the smaller the angle of the wedge becomes, the more easily it exerts its effect” (2.4). With regard to the compound pulley, he notes that, “the more parts the rope is bent into, the easier it is to move the load” (2.3). These general judgments are based on the routine handicraft practice, which does not involve any mathematical knowledge.

Now, the theoretical part of “Mechanics” does radically differ from its practical part as well as from other ancient manuals. In it, Heron makes a unique for his time attempt to quantify the parameters of movements produced by “simple machines.” Their “wonderful property” to move large loads by a small force has for the first time got an explanation in mathematical terms. The main idea of Heron is to derive the relations between the parameters of the motion of “simple machines” from the mathematical law of equilibrium of weights on scales. In doing that, Heron proceeds from the premise that all
kinds of movements occurring in these machines can be reduced to statics of weights described in terms of Archimedes' equilibrium law.

In the examples given by Heron, two dissimilar forces are compared. One of them is the man's living (animated) power, attached to the machine, and the other – the mechanical force of gravity, which resists the movement of the working part of the machine. The former force is active, the latter is passive. Comparing these forces and expressing the result of comparison in terms of the theory of proportion, Heron goes, as it were, against the classical Greek viewpoint, according to which the existence of a mathematical relation between the quantities implies the belonging of these quantities to the same genus. In order to compare the heterogeneous quantities, these quantities must first be brought to a common genus. This is exactly what Heron does, equating the living effort developed by the worker (slave), to a certain weight that can be set in motion by him. Usually, Heron takes this weight to be equal to 5 talents (about 130 kg.). As a result of this identification, both forces become homogeneous. “Transforming” the living effort into weight (the force of gravity) and thus ensuring the homogeneity of forces, Heron conducts his reasoning, focusing on the mathematical model of a simple machine, “at the input “ of which a weight of 5 talents is applied, and “in the output” – a weight which is a multiple of 5 talents.

The problem of the movement of the lever, and of the wheel and the axle was easily reduced by Heron to the static problem of balancing weights on the scales by using the model of two concentric circles. The same holds with regard to single block too.

However, in the case of the other two machines - compound pulley and wedge - the reduction procedure is only declared, but remains unrealized. Although the relation between force and lifted weight (in the case of compound pulley), and the force of blow and the effect achieved (in the case of a wedge) are calculated correctly, Heron does not in fact give an explanation of these facts. In the first case, the analogy between compound pulley and a simple block is purely formal. In the second, there is not even such an analogy. As for the problem of the force moving the ball up the inclined plane, Heron proposes to find it, starting from the equilibrium conditions between the parts of the ball, which are obtained by the section of the ball by a vertical plane passing through the point of its touch with the inclined plane. Here the parallel with the weights on the scales, indeed, can be traced; however, specific calculations for this model are lacking in the treatise. Pappus of Alexandria later tried to solve the same problem, but did not reach a correct solution.

Despite the significance (from the modern point of view) of the quantitative results obtained by Heron, his idea of the mathematical description of “simple machines” was not elaborated. In fact, the Greek text of his “Mechanics” is not even preserved; its content is now known through its Arabic version only. There are no other ancient texts devoted to the analysis of technical motion in mathematical terms. In his “Mathematical Collection” Pappus gives the description of “simple machines,” borrowed from the practical part of the Heron’s “Mechanics,” but the quantitative estimates by Heron are lacking in it. It is possible that the theoretical part of the treatise by Heron went at that time out of an active circulation (Pappus in the introduction to his Book complains about the lack of good manuscripts of “Mechanics”).

This short overview suggests the conclusion that the mathematization of the motion of “simple machines” was not among the priorities of ancient mechanics. The same can be said about the mechanics of the Middle Ages, at least in the Latin West, where the problem of finding exact quantitative relationships between the applied effort and its outcome was not even formulated.

A completely different picture arises when we look into the manuals on practical mechanics compiled in the second half of the 16th century. Unlike their ancient predecessors, the authors of this era strongly emphasize the quantitative aspects of the motion of “simple machines.” This tradition culminated in Galileo’s early treatise
“Mechanics” (about 1593), in which he set out methods that allowed to calculate the coefficients of the transformation of forces and motions in almost all modifications of “simple machines” while demonstrating their mathematical correctness. His calculations were based upon a dynamic version of the rule of lever, which Galileo derived from the law of equilibrium of weights on scales [Galileo, 1964].

It is important to note that the quantitative evaluation of the effectiveness of “simple machines” was as important for a brilliant theorist Galileo, as for his more practically oriented contemporaries. In this connection, Guididaldo del Monte should be mentioned, who was the first technician having performed calculations with respect to “simple machines” (Mechanica, 1577). He influenced not only young Galileo, but also Brunetto Lorini, the author of the famous treatise “On Fortifications” (1593) and Bernardino Baldi, one of the best commentators of “Mechanical Problems” (his Commentary, containing calculations with regard to compound pulley heavily drawn on del Monte, was published in 1621). The beginning of this tradition is associated with Leonardo da Vinci, who performed calculations of simple pulleys already around 1500 (his notation books remained unpublished at the time).

The orientation toward a quantitative result, the variety of calculation techniques applied to a wide range of technical devices, the correct application of the mathematical methods of Archimedes' statics, all this is a sign of the radical transformation occurred in practical mechanics in the second half of the 16th century. The authors of the manuals compiled during this period were no longer satisfied with rough descriptions of movements in terms of “more,” “less,” or “equal”, but strove to supplement them with precise quantitative analysis.

In this connection, the following question arises: what kind of changes occurred in practical mechanics of the 16th century, which provoked the idea of an analysis of technical movement in mathematical terms?

One of the important prerequisites for the progress in the theoretical development of technical movement was a new way of operating lifting mechanisms, based on the use of the force of gravity in a double role – as counterweight and as a lift.

The main feature of the hoisting machines applied in antiquity and the Middle Ages was that they mostly used the muscular power. In some cases it was a joint manual effort, which was applied to ropes, levers, wheels, compound pulleys and other mechanisms for lifting weights. In others, it was a joint effort of the feet, as, for example, in the use of treadwheels driven by people who stepped from step to step.

In practice of this kind, that is, based on animate (animal) power, it was impossible to create a movement that would obey precise numerical ratios. In a joint muscular activity the main task usually consist in the maximization of individual efforts and the implementation of a synchronized action. This is obvious, for example, in the treadwheels, where great efforts were spent to ensure synchronicity of movements. In such situations, the action of forces was subordinated to a qualitative law, according to which, the joint force was greater than the sum of individual efforts of which it consisted. In other words, in such technical movements physical forces could not be considered as extensive magnitudes, since they were, in modern parlance, not additive. The entire force was thought to exceed the sum of its parts [Zaytsev, 2015]. This kind of effect could not be described in terms of mathematics for which all magnitudes are extensive and additive.

Another obstacle for the development the quantitative assessment of forces was the heterogeneity of the forces involved in the movement of machines. As mentioned above, in the hoisting of the time two different types of forces were involved: on the one hand, animate (animals) force which produced motion, and on the other – the mechanical force of gravity which resisted it. A relation between these two forces could be described in terms “greater than,” “less than,” and “equal” only, basing on the fact of produced movement or lack thereof. To establish a mathematical relation between the living and the mechanical
force one needed to carry out a formal transformation of one of them into the other, ascertaining their homogeneity (e.g. by equating the force of a human being to a certain weight as in Heron’s “Mechanics”).

During the 15th–16th c. a substantial novelty occurred in the operation of the lifting machines, which has resulted in the elimination (i) of the effect of “joint force,” and (ii) of the heterogeneity of forces involved in movement. For raising (lowering or simply displacing) the weights, counterweights began to be used. The practical value of this method consisted in that it allowed a heavy load previously balanced by the counterweight be displaced by using a slight force (infinitesimally small compared to the weight of the load). In addition, the application of counterweights made it possible to reduce the size of lifting devices.

A “theoretical” value of hoists with counterweight consisted in that both forces (that which drove the machine and that which resisted its movement) were the forces of gravity, that is, belonged to the same genus. Subsuming forces under a common genus was the key condition that allowed expressing the relation of forces involved in the movement in mathematical terms. In addition, the use of counterweights helped in solving the dynamic problem (which was rather difficult due to irregularities associated with the animate nature of the forces) by analogy with the static problem, the solution of which, based on the law of lever, was simple. The authors of the manual on practical mechanics, written in the 16th c. made a broad application of this method to calculate the gain in strength and, accordingly, loss in speed and distance, occurred in the operation of “simple machines.”

This work was supported by the Russian Foundation for Fundamental Basic Research (project number 15-03-00218a).

Literature